

THE UNIVERSAL HOMOGENEOUS TRIANGLE-FREE GRAPH HAS FINITE RAMSEY DEGREES

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ABSTRACT. The universal homogeneous triangle-free graph \mathcal{H}_3 has finite ‘big Ramsey degrees’: For each finite triangle-free graph G , there is a finite number $n(G)$ such that for any coloring c of all copies of G in \mathcal{H}_3 into finitely many colors, there is a subgraph \mathcal{H}' of \mathcal{H}_3 which is isomorphic to \mathcal{H}_3 in which c takes on no more than $n(G)$ many colors.

The proof hinges on the following developments: a new flexible method for constructing trees which code the universal triangle-free graph, called strong coding trees; a new notion of isomorphism type of finite subtrees of a strong coding tree; analogues of the Halpern-Läuchli and Milliken Theorems, obtaining a Ramsey theorem for strict similarity types of finite subtrees of a given strong coding tree; and new notions of subtree envelope. The proof of the analogue of the Halpern-Läuchli theorem for strong coding trees uses forcing techniques, though the proof is in ZFC, building on ideas from Harrington’s proof of the Halpern-Läuchli Theorem.

Note: There will be an improved draft in the near future, with some graphics of the trees and more exposition if needed. Due to deadlines for other projects, I will not be able to revise this draft for one month, so I am making this draft public to show that the work is completed.

1. INTRODUCTION

It is a central question in the theory of homogeneous relational structures as to which structures have finite big Ramsey degrees. The Fraïssé limit \mathcal{H} of a Fraïssé class \mathcal{K} of finite relational structures is said to have *finite big Ramsey degrees* if for each finite member K in \mathcal{K} , there is a finite number $n(K)$ such that for any coloring c of all the substructures of \mathcal{H} which are isomorphic to K into finitely many colors, there is a substructure \mathcal{H}' of \mathcal{H} which is isomorphic to \mathcal{H} and in which c takes on no more than $n(K)$ colors.

A Fraïssé class and its Fraïssé limit may have quite different behaviors with respect to the Ramsey property. For example, the Fraïssé class of finite ordered graphs has the Ramsey property: For any two finite ordered graphs A and B with A embedding into B and for any finite number k , there is a finite ordered graph C such that for any coloring of the copies of A in C into k colors, there is a copy B' of B in C such that all copies of A in B' have the same color. This theorem, due to Nešetřil and Rödl, is just one case of their broader work

2010 *Mathematics Subject Classification.* 05C15, 03C15, 03E02, 03E05, 03E75, 05C05, 03E45.

This research was commenced and partially done whilst the author was a visiting fellow at the Isaac Newton Institute for Mathematical Sciences in the programme ‘Mathematical, Foundational and Computational Aspects of the Higher Infinite’ (HIF). It was continued while the author was a visitor at the Centre de Recerca Matemàtica in the ‘Intensive Research Program on Large Cardinals and Strong Logics.’ The author gratefully acknowledges support from the Isaac Newton Institute, the Centre de Recerca Matemàtica, and from National Science Foundation Grants DMS-1301665 and DMS-1600781.

in [7] and [8] showing that a large collection of Fraïssé classes of finite ordered relational structures has the Ramsey property.

The Rado graph is the countable universal homogenous graph, and is the Fraïssé limit of the Fraïssé class of countable graphs. It is an easy exercise from the defining property of the Rado graph to show that the Rado graph is *indivisible*: For any coloring of the vertices of the Rado graph into finitely many colors, there is a subgraph \mathcal{R}' which is also a Rado graph in which all vertices have the same color. In contrast, the Rado graph does not have the Ramsey property for colorings finite graphs in general. This phenomenon appears already for edges: There is a coloring of the edges in the Rado graph into two colors such that for any subgraph \mathcal{R}' of the Rado graph such that \mathcal{R}' is also universal for countable graphs, the edges in \mathcal{R}' take on both colors [11]. On the other hand, for any coloring of the edges in the Rado graph into finitely many colors, there is a copy of the Rado graph in which all edges take on no more than two colors [9]. This result was extended by Sauer in [11] to show that all finite graphs have finite Ramsey degrees in the Rado graph, as well as extensions of this result other homogeneous binary relational structures.

This article is concerned with similar phenomena for the universal homogeneous countable graph omitting triangles. A graph G is called *triangle-free* if for any three vertices in G , there is at least one pair with no edge between them; in other words, no triangle embeds into G as a subgraph. A graph \mathcal{H} on countably many vertices is a *universal triangle-free graph* if each triangle-free graph on countably many vertices embeds into \mathcal{H} . Universal triangle-free graphs were first constructed by Henson in [4], and are seen to be the Fraïssé limit of \mathcal{K}_3 , the Fraïssé class of all countable triangle-free graphs. Henson proved that every universal triangle-free graph is homogeneous, and vice versa. A graph H on countably many vertices is *homogeneous* if whenever G is a finite subgraph of H , then every embedding of G into H can be extended to an automorphism of H . Henson showed that any two universal homogeneous triangle-free graphs are isomorphic. It is fitting then to use the notation \mathcal{H}_3 to denote the universal homogeneous triangle-free graph.

Work in [7] and [8] showed that \mathcal{K}_3' , the class of finite ordered triangle-free graphs, has the Ramsey property: For each pair of finite ordered triangle-free graphs A and B such that A embeds into B , and for each finite number k , there is a finite ordered triangle-free graph C such that for each coloring of the copies of A in C into k colors, there is a copy B' of B in C such that the copies of A in B' all have the same color.

In contrast, whether or not the big Ramsey degrees for \mathcal{H}_3 are finite has been open for some time. The first result on colorings of vertices of \mathcal{H}_3 was obtained by Henson in [4] in 1971; there, he proved that given any coloring of the vertices of \mathcal{H}_3 into two colors, either there is a copy of \mathcal{H}_3 in which all vertices have the first color, or else a copy of each member of \mathcal{K}_3 can be found with all vertices having the second color. From this follows a prior result of Folkman in [2] showing that for any finite triangle-free graph G and any number $n \geq 2$, there is a finite triangle-free graph H such that for any partition of the vertices of H into n pieces, there is a copy of G in which has all its vertices in one of the pieces of the partition. Fifteen years later, Komjáth and Rödl proved that \mathcal{H}_3 has the Ramsey property for vertex colorings; that is, they proved in [5] that for any partition of the vertices in \mathcal{H}_3 into finitely many pieces, there is subgraph \mathcal{H}' of \mathcal{H}_3 such that the vertices in \mathcal{H}' lie in exactly one of the pieces of the partition. It then became of interest whether this result would extend to colorings of copies of a fixed finite triangle-free graph, rather than just colorings of vertices.

Definition 1 (Finite Ramsey Degrees). Let G be a finite triangle-free graph. We say that G has *finite big Ramsey degree* if there is a finite number n , possibly depending on G , such that for each coloring of all the copies of G in \mathcal{H}_3 into finitely many colors, there is a subgraph \mathcal{H}' of \mathcal{H}_3 which is also universal triangle-free and such that all copies of G in \mathcal{H}' take on no more than n colors. In this case, we let $n(G)$ denote the least number such that all such colorings into finitely many colors can be restricted to $n(G)$ many colors on some copy of \mathcal{H}_3 . We then say that $n(G)$ is *the big Ramsey degree of G* .

We say that \mathcal{H}_3 has *finite big Ramsey degrees* if every finite triangle-free graph has finite Ramsey degree.

In 1998, Sauer proved in [10] that edges have finite big Ramsey degree of 2 in \mathcal{H}_3 . This left open the following question:

Question 2. Does every finite triangle-free graph have finite Ramsey degree in \mathcal{H}_3 ?

This paper answers this question in the affirmative.

Sauer proved in 2006 that the Rado graph has finite big Ramsey degrees. The outline of Sauer's proof in [11] provided a strategy for our proof in this paper. A rough outline of Sauer's proof is as follows: Graphs can be coded by nodes on trees. Given such codings, the graph coded by the nodes in $2^{<\omega}$ is bi-embeddable with the Rado graph. Only certain subtrees, called strongly diagonal, are necessary to consider, when handling tree codings of a given finite graph G . Any finite strongly diagonal tree can be enveloped into a strong tree, which is a tree isomorphic to $2^{\leq k}$ for some k . The coloring on the copies of G can be extended to color the strong tree envelopes. Applying Milliken's Theorem for strong trees [6] finitely many times, one finds a strong subtree S of $2^{<\omega}$ in which every similarity type coding G has the same color. Then all copies of G in S have no more colors than the number of similarity types. To finish, take a strongly diagonal D subtree of S which codes the Rado graph. Then all subtrees of D coding G are strongly diagonal, and hence have one color per similarity type. Since there are only finitely many similarity types, this yields the finite Ramsey degrees for the Rado graph.

This outline seemed to us the most likely to succeed if indeed the universal triangle-free graph were to have finite Ramsey degrees. However, there were difficulties involved in each step of trying to adapt Sauer's proof to the setting of \mathcal{H}_3 , largely because \mathcal{H}_3 is a very different type of homogeneous structure as it omits a substructure, namely triangles. First, unlike the bi-embeddability between the Rado graph and the graph coded by $2^{<\omega}$, there is no bi-embeddability relationship between \mathcal{H}_3 and some other triangle-free graph coded by some tree with a simple structure. To handle this, rather than letting certain nodes in a tree code vertices at the very end of the whole proof scheme, we defined a new notion of *strong triangle-free tree* in which we distinguish certain nodes *in* the tree to code the vertices of a given graph (called *coding nodes*), and in which the branching is maximal subject to the constraint of these distinguished nodes not coding any triangles. We further developed a flexible construction method for creating strong triangle-free trees in which the distinguished nodes code \mathcal{H}_3 . These are found in Section 2.

Next, we wanted an analogue of Milliken's Theorem for strong subtrees. While we were able to prove this in strong triangle-free trees (the type of tree defined in Section 2) for all configurations extending some fixed stem, the result simply did not hold for colorings of stems, as can be seen by an example of a bad coloring defined using interference between

splitting nodes and coding nodes on the same level. The way around this was to define a notion of *strong coding tree*, which in particular are skew trees that stretch the structure of strong triangle-free trees. Strong coding trees are defined in Section 3. The analogue of the Halpern-Läuchli Theorem ([3]) for strong coding trees, where the level being colored contains a splitting node, is proved in Section 4. A similar result is proved for level sets containing a coding node, obtaining end-homogeneity. However, more work is needed to homogenize over all extensions, and that takes place later in Section 5. The proof of the main theorem of Section 4 uses the set-theoretic method of forcing, but produces a result using only the axioms used in most standard mathematics, ZFC. We point out that it follows from the main theorem of 4 and some standard checking that the collection of all subtrees of T which are isomorphic to T satisfy all four of Todorčević's axioms in [12] guaranteeing a topological Ramsey space, except for Axiom **A.3(2)**.

In Section 5, we obtain an analogue of Milliken's Theorem for finite trees satisfying a property we call Strict Parallel 1's Criterion. In order to ensure that we can find the precise big Ramsey degrees, more developments were needed. The development of the notions of strong similarity type for all diagonal trees and strong subtree envelope for the setting of triangle-free graphs presented a further obstacle. Our version of Milliken's Theorem in Section 5 applies only to triangle-free trees with a certain property called the Strict Parallel 1's Criterion, as these are the types of trees for which the related forcing can still produce a strong coding tree coding \mathcal{H}_3 in which all strictly similar copies of a given finite tree have one color.

The idea of extending a triangle-free tree to an envelope which is a finite strong triangle-free tree (which would be the direct analogue of Sauer's method) simply does not work, as it can lead to an infinite regression of coding nodes to add. To overcome this, in Sections 6 and 7 we develop new notions of incremental new parallel 1's and strict similarity type for finite diagonal trees as well as a new notion of subtree envelope. Given any finite triangle-free graph G , there are only finitely many strict similarity types of diagonal trees coding G . Letting c be any coloring of all copies of G in \mathcal{H}_3 into finitely many colors, we transfer the coloring to the subtree envelopes and apply the results in previous sections to obtain a strong coding tree S in which all subtree envelopes encompassing the same strong similarity type have the same color. After thinning to a strongly diagonal tree $D \subseteq S$ still coding \mathcal{H}_3 , we find that each subtree of D with the same strict similarity type coding G has the same color. This then finishes the proof that the universal homogeneous triangle-free graph has finite Ramsey degrees. The number $n(G)$ may be obtained by the patient observer willing to count numbers of strict similarity types of incremental strongly diagonal trees.

Acknowledgments. Much gratitude goes to Dana Bartořová for listening to and making helpful comments on early and later stages of these results and for her continued encouragement of my work on this problem; Jean Larson for listening to early stages of this work and her encouragement; Norbert Sauer for discussing key aspects of the homogeneous triangle-free graph with me during a research visit in Calgary in 2014; Stevo Todorčević for pointing out to me in 2012 that any proof of finite Ramsey degrees for \mathcal{H}_3 would likely involve a new type of Halpern-Läuchli Theorem; and to the organizers and participants of the Ramsey Theory Doccourse at Charles University, Prague, 2016, for their encouragement. Most of all, I am grateful for and much indebted to Richard Laver for providing for me in 2011 the main points of Harrington's forcing proof of the Halpern-Läuchli Theorem, from which I could

reconstruct the proof, setting the stage for the possibility of accomplishing this work. His spirit lives on.

2. STRONG TRIANGLE-FREE TREES CODING \mathcal{H}_3

As is well-known, in particular from [9], [10] and [11], any countable graph G may be represented by nodes in a subtree of $2^{<\omega}$ as follows. Given nodes $s, t \in 2^{<\omega}$ representing vertices v, x , if the length of s is less than the length of t as sequences of 0's and 1's, denoted as $|s| < |t|$, then the pair s, t codes an edge between v and x if and only if $t(|s|) = 1$. The number $t(|s|)$ is called the *passing number*. Thus, a passing number of 1 codes an edge and a passing number of 0 codes a non-edge. Let $\langle v_n : n < N \rangle$ be any enumeration of the vertices of G , where $N \leq \omega$ is the number of vertices in G . Choose any node $t_0 \in 2^{<\omega}$ to represent the vertex v_0 . Let $l_0 = |t_0|$, the length of t_0 as a sequence of 0's and 1's. For $n > 0$, given nodes t_0, \dots, t_{n-1} in $2^{<\omega}$ coding the vertices v_0, \dots, v_{n-1} , with l_{n-1} denoting $|t_{n-1}|$, let t_n be any node in $2^{<\omega}$ such that $|t_n| > l_{n-1}$ and satisfying that for all $i < n$, v_n and v_i have an edge between them if and only if $t_n(l_i) = 1$. The set $\{t_n : n < N\}$ codes the graph G . We shall call the nodes t_n *coding nodes* in the tree representation of G , for they are the nodes coding the vertices in the graph G . Note that any finite graph of size k can be coded by a collection of coding nodes in $2^{<k}$. Throughout this paper it is always the case that any two coding nodes have different lengths.

Definition 3. A graph G with vertices enumerated as $\langle v_n : n < N \rangle$ is *represented* by a tree T with coding nodes $\langle t_n : n < N \rangle$ if and only if for each pair $i < n < N$, $|t_i| < |t_n|$ and $v_n E v_i \iff t_n(l_i) = 1$. We will often simply say that T *codes* G .

Next, we look at which tree configurations code triangles. These will be the configurations that must be omitted from any tree coding of a triangle-free graph. Note that if v_0, v_1, v_2 are vertices with edges between each pair and t_0, t_1, t_2 are coding nodes coding the vertices v_0, v_1, v_2 and the edge relationships between them, with $|t_0| < |t_1| < |t_2|$, then it must be the case that $t_1(|t_0|) = t_2(|t_0|) = t_2(|t_1|)$. Moreover, this is the only way a triangle can be coded.

We now present a criterion which, if satisfied, guarantees that any maximal node in a tree may be extended to a coding node without coding a triangle.

Definition 4 (Triangle-Free Criterion). Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle t_n : n < N \rangle$, where $N \leq \omega$. T *satisfies the Triangle-Free Criterion (TFC)* if the following holds: For each $s \in T$, if $l_n < |s|$ and $s(l_i) = t_n(l_i) = 1$ for some $i < n$, then $s(l_n) = 0$.

In words, a tree T with coding nodes satisfies the TFC if for each $n < N$, whenever a node u in T has the same length as coding node t_n , and u and t_n both have passing number 1 at the level of coding node t_i for some $i < n$, then $u \frown 1$ must not be in T . Note that if T is a finite tree satisfying the TFC, then any maximal node of T may be extended to a coding node without coding a triangle with the coding nodes in T .

Eventually, we will want to code \mathcal{H}_3 by coding nodes in some tree $\mathbb{S} \subseteq 2^{<\omega}$, and we will want these coding nodes to be dense in \mathbb{S} . Thus, in our construction of \mathbb{S} , the finite initial subtrees of \mathbb{S} will need to satisfy this criterion in order to be extendible to trees which code \mathcal{H}_3 and in which the coding nodes are dense. Such a tree \mathbb{S} will be constructed in Theorem 10.

For $s, t \in 2^{<\omega}$, we use the following standard notation: $s \subseteq t$ denotes that s is an initial segment of t , and $s \subset t$ denotes that s is a proper initial segment of t . The next lemma provides a characterization of tree representations of triangle-free graphs.

Lemma 5 (Triangle-Free Tree Representation). *Let $T \subseteq 2^{<\omega}$ be a tree representation of a countable graph G with coding nodes $\{t_k : k < N\}$, where each level of T has at most one coding node. Assume that the coding nodes in T are dense in T ; that is, for each $u \in T$, there is some coding node $t_k \in T$ such that $u \subseteq t_k$. Then the following are equivalent:*

- (1) G is triangle-free.
- (2) T satisfies the Triangle-Free Criterion.

Proof. Let $N \leq \omega$ denote the number of vertices of the graph G . We point out that if N is finite, then the coding nodes in T being dense in T means that every maximal node in T is a coding node; in this case, the maximal nodes in T have different lengths. For each $k < N$, let l_k denote $|t_k|$.

Suppose (2) fails. Then there are $i < j < N$ and $s \in T$ with length greater than l_n such that $s(l_i) = t_j(l_i) = 1$ and $s(l_j) = 1$. Since every node in T extends to a coding node, there is an $k > j$ such that $t_k \supseteq s$. Then $t_k(l_j) = 1$ and $t_k(l_i) = s(l_i) = 1$. Thus, the vertices $\{v_i, v_j, v_k\}$ have edges between each pair, implying G contains a triangle. Therefore, (1) fails.

Conversely, suppose that (1) fails. Then G contains a triangle, so there are $i < j < k$ such that the vertices v_i, v_j, v_k have edges between each pair. Note that $v_j E v_i$ and $v_k E v_i$ imply $t_j(l_i) = t_k(l_i) = 1$. Further, $v_k E v_j$ implies $t_k(l_j) = 1$. Hence, letting $s = t_k \upharpoonright l_j$, we have $s \in T \cap 2^{l_j}$ and $s(l_i) = t_j(l_i) = 1$. Further, $s \frown 1$ is in T , since t_k is in T , $s \frown 1 \subseteq t_k$, and T is equal to the collection of all initial segments of its coding nodes. Hence, (2) fails. \square

Sauer's proof in [11] that the Rado graph has finite Ramsey degrees uses the fact that the Rado graph is bi-embeddable with the graph coded by all nodes in $2^{<\omega}$, where two nodes of the same level do not code an edge between the vertices associated with those nodes. Colorings on the Rado graph are transferred to the graph represented by $2^{<\omega}$ where Milliken's Theorem for strong trees is applied, and then the homogeneity is transferred back to the Rado graph. In the case of the universal triangle-free graph, there is no such bi-embeddability between \mathcal{H}_3 and some triangle-free graph coded by nodes in a tree with some kind of uniform structure. To handle this situation, we instead define trees with some of the nodes in them being distinguished to code vertices in triangle-free graphs, and then let them branch maximally subject to not coding any triangles. These trees then are analogues of Milliken's strong trees and hence are called *strong triangle-free trees*.

We now define strong triangle-free trees and present a flexible construction method for obtaining trees coding \mathcal{H}_3 . These trees will not be sufficient for proving a Milliken-style theorem: though such a theorem holds above any fixed stem, it does not hold at the very basic level of colorings of stems of strong triangle-free trees, as there is a 'bad' coloring witnessing this. However, upon forming diagonalized variants of these trees (see Section 3), then a Milliken-style theorem will indeed hold (see Section 7). The present section is intended to develop the reader's understanding of strong triangle-free trees before defining the skew variation in the Section 3.

Definition 6. A countably infinite graph \mathcal{G} with vertices enumerated as $\langle v_n : n < \omega \rangle$ is *densely represented* by the coding nodes of a tree T if and only if T represents \mathcal{G} and the coding nodes are dense in T .

Remark. If T densely represents a graph \mathcal{G} , then T is equal to the closure under initial segments of its set of coding nodes.

Definition 7 (Strong triangle-free tree). A tree $T \subseteq 2^{<\omega}$ with coding nodes is a *strong triangle-free tree* if T satisfies the Triangle-Free Criterion and each node in T is maximally splitting, subject to satisfying the TFC.

T is a *strong triangle-free tree densely coding* \mathcal{H}_3 if T is a strong triangle-free tree and \mathcal{H}_3 is densely represented by the coding nodes in T .

We now present a method for constructing strong triangle-free trees densely coding \mathcal{H}_3 with the additional property that every coding node does not split. This can be done by requiring that we order the vertices of \mathcal{H}_3 in such a way that each vertex v_{i+1} has an edge with v_i . The tree representation here is defined in such a way as to make density of the coding nodes possible by a simply indexed construction.

Let \mathcal{K}_3 denote the Fraïssé class of all triangle-free countable graphs. In [4], Henson proved that a countable graph H is universal for \mathcal{K}_3 if and only if H satisfies the following property:

- (A₃) (i) H does not admit any triangles,
- (ii) If V_0, V_1 are disjoint, finite sets of vertices of H and $H|V_0$ does not admit an edge, then there is another vertex which is connected in H to every member of V_0 and to no member of V_1 .

Henson used this property to construct a universal triangle-free graph \mathcal{H}_3 in [4], as well as universal graphs for each Fraïssé class of countable graphs omitting k -cliques, as the analogues of the Rado graph for countable k -clique free graphs. The following property (A₃)' is a formulation of Henson's property (A₃).

- (A₃)' (i) H does not admit any triangles,
- (ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of $[\omega]^{<\omega}$ such that for each $i < \omega$, $\max(F_i) < i$ and each finite set is repeated infinitely often; that is, for each $B \in [\omega]^{<\omega}$, the set $\{n < \omega : F_i = B\}$ is infinite. Then there is a subsequence $\langle k_i : i < \omega \rangle$ such that for all $i < \omega$, if $H|\{v_m : m \in F_i\}$ is edge free then for all $m < i$, $v_{k_i} E v_m \longleftrightarrow m \in F_i$.

It is straightforward to check the following fact.

Fact 8. Let H be a countably infinite graph with vertices enumerated as $\langle v_i : i < \omega \rangle$. Then H is universal for \mathcal{K}_3 if and only if (A₃)' holds.

The following tree re-formulation of property (A₃)' will be used throughout this article. Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\{t_k : k < \omega\}$, and let l_k denote $|c_k|$. We say that T satisfies property (A₃)^{tree} if the following holds:

- (A₃)^{tree} (i) T satisfies the Triangle-Free Criterion,
- (ii) Let $\langle F_i : i < \omega \rangle$ be any listing of finite subsets of ω such that $F_i \subseteq i$ and each finite set appears as F_i for infinitely many indices i . For each $i < \omega$, if $t_k(l_j) = 0$ for all pairs $j < k$ in F_i , then there is some $n \geq i$ such that for all $k < i$, $t_n(l_k) = 1$ if and only if $k \in F_i$.

Fact 9. Suppose $T \subseteq 2^{<\omega}$ is a tree with coding nodes $\{t_k : k < \omega\}$, where there is at most one coding node per level. Then T codes \mathcal{H}_3 if and only if T satisfies (A₃)^{tree}.

We now present a construction method for building strong triangle-free trees densely coding \mathcal{H}_3 with the additional property that no coding node in the tree splits.

Theorem 10 (Strong Triangle-Free Tree \mathbb{S} Densely Coding \mathcal{H}_3). *Let $\langle F_i : i < \omega \rangle$ be any sequence enumerating the finite subsets of ω in such a manner that each finite set appears cofinally often and for each $i < \omega$, $\max(F_i) \leq i - 2$ and moreover $F_{3i} = F_{3i+2} = \emptyset$. Then there is a strong triangle-free tree \mathbb{S} which densely codes \mathcal{H}_3 , where the coding node c_{4i+j} takes care of requirement F_{3i+j} , for $i < \omega$ and $j < 3$. Furthermore, the tree \mathbb{S} will code \mathcal{H}_3 in such a manner that for all $n < \omega$, $v_{n+1} \text{ E } v_n$.*

Proof. Let $\langle F_i : i < \omega \rangle$ satisfy the hypothesis, and let $\langle u_q : q < \omega \rangle$ be an enumeration of all the nodes in $2^{<\omega}$ so that $|u_q| \leq q$ for each $q < \omega$. We will build a strong triangle-free tree $\mathbb{S} \subseteq 2^{<\omega}$ with coding nodes $c_n \in \mathbb{S} \cap 2^{n+1}$ densely coding \mathcal{H}_3 satisfying the following properties:

- (1) \mathbb{S} has no terminal nodes.
- (2) For $n = 4i + j$, $j < 3$, c_n satisfies requirement F_{3i+j} .
- (3) For $n = 4i + 3$, if u_i is in \mathbb{S} then c_n is a coding node extending u_i . If u_i is not in \mathbb{S} , then $c_n = 0^n \hat{\ } 1$.
- (4) For each $n < \omega$, $c_n \hat{\ } 1 \notin \mathbb{S}$ and $c_{n+1}(|c_n|) = 1$.

To begin, let $\mathbb{S} \cap 2^{\leq 1} = 2^{\leq 1}$ and let $c_0 = \langle 1 \rangle$. Then c_0 takes care of requirement F_0 since $F_0 = \emptyset$. Let $\mathbb{S} \cap 2^2 = 2^2$, as this is maximally splitting above $\mathbb{S} \cap 2^{\leq 1}$ subject to satisfying the TFC with respect to the coding nodes in $\mathbb{S} \cap 2^{\leq 1}$.

For the general construction step, suppose we have constructed $\mathbb{S} \cap 2^{\leq n+1}$ and chosen coding nodes c_j , $j < n$, such that for each $1 \leq k \leq n$, $\mathbb{S} \cap 2^{k+1}$ is maximally branching over $\mathbb{S} \cap 2^k$ subject to satisfying the TFC with respect to $\mathbb{S} \cap 2^{\leq k}$. There are three cases for how to choose c_n in $\mathbb{S} \cap 2^{n+1}$.

Case 1. $n = 4i$ and $i \geq 1$, or $n = 4i + 2$ and $i < \omega$. Let n' denote $3i$ if $n = 4i$, and let n' denote $3i + 2$ if $n = 4i + 2$. Then $F_{n'} = \emptyset$, so let $c_n = 0^n \hat{\ } 1$.

Case 2. $n = 4i + 1$ and $i < \omega$. If for all pairs $k < m$ of integers in F_{3i+1} , $c_m(|c_k|) = 0$, then take c_n to be the node in $\mathbb{S} \cap 2^{n+1}$ such that for all $k < n$, $c_n(|c_k|) = 1$ if and only if $k \in F_{3i+1}$, and $c_n(|c_n|) = 1$.

Case 3. $n = 4i + 3$ and $i < \omega$. Note that $u_i \in 2^{\leq i}$ and $i \leq n - 3$. If $u_i \in \mathbb{S} \cap 2^{\leq i}$, then take $c_n \in \mathbb{S} \cap 2^{n+1}$ to be u_i extended by all 0's except at the top, where it is extended by 1. Precisely, letting $q = n - |u_i|$, take $c_n = u_i \hat{\ } 0^q \hat{\ } 1$.

Notice that for all $k \leq n$, the node $0^k \hat{\ } 0$ is in $\mathbb{S} \cap 2^{\leq n+1}$, since 0^k has no parallel 1's with any coding node in $\mathbb{S} \cap 2^{\leq n}$. In Case 1, the maximality of the branching in $\mathbb{S} \cap 2^{n+1}$ over $\mathbb{S} \cap 2^{\leq n}$ subject to the TFC guarantees that the node we chose as c_n is in $\mathbb{S} \cap 2^{n+1}$. In Case 2, if for all pairs $k < m$ of integers in F_{3i+1} , $c_m(|c_k|) = 0$, then by maximality of branching subject to the TFC, there is a node $s \in \mathbb{S} \cap 2^{n+1}$ such that for each $k < n$, $s(|c_k|) = 1$ if and only if $k \in F_n$; and this is the node we take to be c_n . In Case 3, the fact that $c_{n-1} = 0^n \hat{\ } 1$ and $|u_i| \leq i < n$ implies that the node $u_i \hat{\ } 0^q$ has no parallel 1's with c_{n-1} . Hence, by the maximality of branching of $\mathbb{S} \cap 2^{\leq n}$ with respect to the TFC, $u_i \hat{\ } 0^q$ splits to $u_i \hat{\ } 0^q \hat{\ } 0$ and $u_i \hat{\ } 0^q \hat{\ } 1$ in $\mathbb{S} \cap 2^{n+1}$.

Now that the coding node c_n has been defined to be a member of $\mathbb{S} \cap 2^{n+1}$, $\mathbb{S} \cap 2^{n+2}$ is determined by taking maximal branching above $\mathbb{S} \cap 2^{\leq n+1}$ with the coding nodes $\{c_k : k \leq n\}$

subject to satisfying the TFC. Note that the coding nodes c_{4i+3} , $i < \omega$, guarantee that the coding nodes in \mathbb{S} are dense in \mathbb{S} .

This completes the construction of \mathbb{S} , a prototypical strong triangle-free tree in which the coding nodes are dense and code \mathcal{H}_3 . \square

The previous construction can be carried out for any sequence $\langle F_i : i < \omega \rangle$ enumerating the finite subsets of ω , repeating each set infinitely many times.

The following key properties of the construction are pointed out. Notice that $0^{<\omega} \subseteq \mathbb{S}$, and for each $n < \omega$, $0^n \frown 1 \in \mathbb{S}$. For each node t in $\mathbb{S} \cap 2^n$, $t \frown 0$ is in $\mathbb{S} \cap 2^{n+1}$. For each node $t \in \mathbb{S} \cap 2^{n+1}$, $t \frown 1$ is in $\mathbb{S} \cap 2^{n+2}$ if and only if t and c_n have no parallel 1's; that is, if and only if for all $k < n$, $t(|c_k|)$ and $c_n(|c_k|)$ are not both simultaneously 1. Given any $n < \omega$ and $A \subseteq n$ such that for all pairs $j < k$ in A , $c_k(|c_j|) = 0$, there is a $s \in \mathbb{S} \cap 2^{n+1}$ such that for all $j \leq n$, $s(|c_j|) = 1$ if and only if $j \in A$.

Remark. The purpose of constructing strong triangle-free trees with coding nodes which do not split is to simplify later work, reducing the number of different isomorphism types of trees coding a given finite triangle-free graph. The purpose of the density of the coding nodes is to allow to thin to subtrees which still can code \mathcal{H}_3 .

3. STRONG CODING TREES

One of the main goals of this paper is to prove an analogue of the Halpern-Läuchli Theorem in the setting of trees with coding nodes coding the universal triangle-free graph \mathcal{H}_3 . In order to obtain such a theorem, it is necessary to work with skewed versions of the strong triangle-free trees from the previous section. In this section, we define skew strong triangle-free trees and show that they can densely code \mathcal{H}_3 . Such trees will be called *strong coding trees*. The collection of all subtrees of a strong coding tree T which are isomorphic to T forms a space of trees coding \mathcal{H}_3 . In particular, given a finite strong triangle-free tree A inside a strong coding tree T , there is a flexible method for end-extending A inside T to another strong coding tree. The flexibility of the extension is what will allow for proving Ramsey theorems on strong coding trees. For those familiar with Todorcevic's axioms guaranteeing a topological Ramsey space (see [12]), we point out that our space of strong coding trees satisfies all four of his axioms except for the second part of Axiom **A.3**.

The following terminology and notation will be used throughout the paper.

Definition 11 (Parallel 1's). For two nodes $s, t \in 2^{<\omega}$, we say that s and t *have parallel 1's* if there is some $l < \min(|s|, |t|)$ such that $s(l) = t(l) = 1$.

For s and t in a tree T with coding nodes, we say that s and t *have parallel 1's at the level of a coding node in T* if there is some coding node c in T for which $s(|c|) = t(|c|) = 1$.

We used standard notation for trees, including $s \wedge t$ to denote the *meet* of s and t ; that is, the longest initial segment they both have in common. As a particular case of Definition 11, for s and t such that $|s \wedge t| < \min(|s|, |t|)$, if there is some $l < |s \wedge t|$ for which $s(l) = t(l) = 1$, then s and t are considered to have parallel 1's.

For $s \in T$ we say that s is a *splitting node* in T if both $s \frown 0$ and $s \frown 1$ have extensions in T . As a convention, we let c_n^T denote the n -th coding node in T , where the coding nodes of T , $\langle c_n^T : n < \omega \rangle$, are listed in order of increasing length. If the tree in question is clear, then we may simply write c_n . We let l_n^T denote $|c_n^T|$, the length of the n -th coding node, and write l_n if the tree T is clear from the context.

The set of *levels* of a tree $T \subseteq 2^{<\omega}$ is the set of those $l < \omega$ such that T has nodes of length l . Between levels of coding nodes, there may be splitting nodes. We point out that the set of levels of a tree T with coding nodes is

$$(1) \quad L^T = L = \{|t| : t \text{ is a coding node or a splitting node in } T\}.$$

The following Parallel 1's Criterion will be used to guarantee that a finite subtree of a strong coding tree T can be extended inside T to have maximal branching subject to the TFC. In particular, if a finite subtree A of T satisfies the Parallel 1's Criterion and is maximally splitting inside T , then it will be possible to end-extend A inside T to another strong coding tree. Thus, the Parallel 1's Criterion could also be called the Strong Coding Tree Extension Guarantee Criterion, but this is longer and doesn't really clarify what is going on structurally.

Given a tree $T \subseteq 2^{<\omega}$, let \hat{T} denote the collection of all initial segments of nodes in T . Thus, whenever $t \in T$, then $t \restriction n$ is in \hat{T} , for all $n < |t|$. Given a node s in T for which there is an initial segment $u \subseteq s$ such that $u \restriction 0$ and $u \restriction 1$ are both in \hat{T} , let $\text{splitpred}_T(s)$ denote the proper initial segment $u \subseteq s$ of maximum length such that $u \restriction 0$ and $u \restriction 1$ are both in \hat{T} . That is, $\text{splitpred}_T(s)$ is the longest proper initial segment of s which is a splitting node in T .

Definition 12 (Parallel 1's Criterion). Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle d_n : n < N \rangle$, for some $N \leq \omega$. We say that T satisfies the *Parallel 1's Criterion* if given any pair $s, t \in T$, for which $s(l) = t(l) = 1$ for some l , the following hold:

- (a) There is an $n < N$ such that $|c_n| < \min(|s|, |t|)$ and $s(c_n) = t(c_n) = 1$;
 - (b) Letting n be least such that $s(c_n) = t(c_n) = 1$, s and t have no parallel 1's below
- $$(2) \quad \max\{|\text{splitpred}_T(u \restriction |c_n|)| : u \in T \restriction |c_n| \text{ and } |c_{n-1}| < |\text{splitpred}_T(u \restriction |c_n|)|\}.$$

Whenever n is least such that $s(c_n) = t(c_n) = 1$, we shall say that the coding node c_n *witnesses* the parallel 1's of s and t .

The following definitions are taken from [11]. Given a subset $S \subseteq 2^{<\omega}$, let S^\wedge denote the set of all meets of pairs of nodes in S ; that is, $S^\wedge = \{s \wedge t : s, t \in S\}$. In this definition s and t may be equal, so S^\wedge contains S .

Definition 13. $S \subseteq 2^{<\omega}$ is an *antichain* if $s \subseteq t$ implies $s = t$, for all $s, t \in S$. A set $S \subseteq 2^{<\omega}$ is *transversal* if $|s| = |t|$ implies $s = t$ for all $s, t \in S$. A set $D \subseteq 2^{<\omega}$ is *diagonal* if D is an antichain and D^\wedge is transversal. A diagonal set D is *strongly diagonal* if additionally for any $s, t, u \in D$ with $s \neq t$, if $|s \wedge t| < |u|$ and $s \wedge t \not\subseteq u$, then $u(|s \wedge t|) = 0$. For $s, t \in 2^{<\omega}$, write $s \prec t$ if s and t are incomparable under \subseteq and $s(l) < t(l)$, where $l = |s \wedge t|$.

Remark. The finite trees A we will be coloring in order to show that \mathcal{H}_3 has finite big Ramsey degrees will be finite trees such that the coding nodes of A are exactly the maximal nodes of A , the set of coding nodes of A is strongly diagonal, and A satisfies the Parallel 1's Criterion.

The following augments Sauer's Definition 3.1 in [11] to the setting of triangle-free trees with coding nodes.

Definition 14. Let $S, T \subseteq 2^{<\omega}$ be trees with certain nodes distinguished as coding nodes. The function $f : S \rightarrow T$ is a *strong similarity* of S to T if for all nodes $s, t, u, v \in S$, the following hold:

- (1) f is a bijection.
- (2) f preserves lexicographic order.
- (3) (f preserves initial segments) $s \wedge t \subseteq u \wedge v$ if and only if $f(s) \wedge f(t) \subseteq f(u) \wedge f(v)$.
- (4) (f preserves meets) $f(s \wedge t) = f(s) \wedge f(t)$.
- (5) (f preserves relative lengths) $|s \wedge t| < |u \wedge v|$ if and only if $|f(s) \wedge f(t)| < |f(u) \wedge f(v)|$.
- (6) (f preserves coding nodes) f maps the set of coding nodes in S onto the set of coding nodes in T .
- (7) (f preserves passing numbers at coding nodes) If c is a coding node in S and u is a node in S with $|u| > |c|$, then $f(u)(|f(c)|) = u(|c|)$.

In all cases above, s and t may be taken to be equal so that $s \wedge t = s$. It follows from (4) that $s \in S$ is a splitting node in S if and only if $f(s)$ is a splitting node in T .

We say that S and T are *strongly similar* if there is a strong similarity of S to T , and in this case write $S \stackrel{s}{\sim} T$. If $T' \subseteq T$ and f is a strong similarity of S to T' , then f is a *strong similarity embedding* of S into T and T' is a *strong similarity copy* of S in T . If $A \subseteq S$, then $\text{Sims}_S^s(A)$ denotes the set of all subtrees of S which are strongly similar to A .

Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n^T : n < \omega \rangle$ listed in order of increasing length. Let l_n denote $|c_n^T|$. The levels of T are $T \cap 2^l$ where l is the length of the stem of T or else the length of a coding node or a splitting node in T . Letting $\langle p_m : m < \omega \rangle$ enumerate in increasing order the lengths of nodes in T , the m -th level of T is $\text{Lev}_T(m) = \{s \in T : |s| = p_m\}$. Let m_n denote the level of T which contains c_n^T ; thus every node in $\text{Lev}_T(m_n)$ has length l_n . For $n \geq 1$, let $\text{Spl}(T, n)$ denote the set of those $t \in \hat{T} \cap 2^{l_{n-1}+1}$ such that there is an extension $s_t \supseteq t$ with $s_t \in \text{Lev}_T(l)$ for some $l \in (m_{n-1}, m_n)$ such that s_t is a splitting node in T ; that is, both $s_t \cap 0$ and $s_t \cap 1$ are in T . For $n = 0$, let $\text{Spl}(T, 0)$ denote the set of those $t \in \hat{T}$ of length $|\text{stem}(T)| + 1$ for which t splits before $\text{Lev}_T(m_0)$. For $t \in \text{Spl}(T, n)$, let $\text{spl}(t)$ denote the minimal extension of t which splits in T . Thus, for $t \in \text{Spl}(T, n)$, $\text{spl}(t) \supseteq t$ and $|\text{spl}(t)| < l_n$. Recall that for $s, t \in 2^{<\omega}$ with neither end-extending the other, $s <_{\text{lex}} t$ if $s \supseteq (s \wedge t) \cap 0$ and $t \supseteq (s \wedge t) \cap 1$.

Definition 15 (Strongly coding tree). $T \subseteq 2^{<\omega}$ is a *strongly coding tree* if the following hold:

- (1) Every level of T contains a member of $0^{<\omega}$, T is infinite and has no maximal nodes, and the coding nodes of T are dense in T .
- (2) (Skew) Each level of T has exactly one of either a coding node or a splitting node.
- (3) (Strongly skew) For each splitting node $s \in T$, every t such that $|t| > |s|$ and $t \not\supseteq s$ also satisfies $t(|s|) = 0$. (That is, the passing number of any node passing by and not extending a splitting node is 0.)
- (4) T satisfies the Triangle-Free Criterion.
- (5) T is maximally splitting subject to satisfying the TFC: For $n \geq 1$ and $s \in T$ of length $l_{n-1} + 1$, s is in $\text{Spl}(T, n)$ if and only if for all $i < n$, $s(l_i) + c_n^T(l_i) \leq 1$.
- (6) T satisfies the Parallel 1's Criterion.
- (7) (Extensions between levels with coding nodes) Each node in $\text{Spl}(T, n)$ has exactly two extensions in $\text{Lev}_T(m_n)$, and each node in $T \cap 2^{l_{n-1}+1} \setminus \text{Spl}(T, n)$ has exactly one extension in $\text{Lev}_T(m_n)$.
- (8) (Passing numbers at levels of coding nodes.) Each node t in $T \cap 2^{l_{n-1}+1} \setminus \text{Spl}(T, n)$ has exactly one extension, say t_* of length $l_n + 1$ in T , and $t_*(l_n) = 0$. For each

$s \in \text{Spl}(T, n)$ and each $i < 2$, there is exactly one extension $s_i \supseteq \text{spl}(t) \frown i$ of length $l_n + 1$ in T , and $s_i(l_n) = i$.

Remark. The meet closure of any antichain of coding nodes in a strong coding tree is strongly diagonal.

Fact 16. *Any strong coding tree is a perfect tree.*

Proof. By (1) in Definition 15, the coding nodes are dense in T . Let t be any node in T and let m denote the level of t in T , that is, $t \in \text{Lev}_T(m)$. Let s denote the leftmost member of $\text{Lev}_T(m)$, so that s is a sequence of 0's. Let m_n be least such that $m_n > m$. Then c_n^T is the least coding node in T above level m . Let $s_* = 0^{l_n}$. Then $s_* \in \text{Lev}_T(m_n)$ and $s_* \supset s$. Let t_* be t extended by all 0's.

Let $p > n$ be least such that $c_p^T \supset s_*$. Let u be the leftmost extension of t_* in $\text{Lev}_T(m_{p-1})$, at the level of the $p - 1$ -st coding node, and let $v = u \frown 0$. Note that v is in \hat{T} , since between levels l_{p-2} and l_{p-1} , u extended $t_* \cap 2^{l_{p-2}}$ via its leftmost extension in T . Hence, for all $i < p$, $v(l_i) + c_p^T(l_i) \leq 1$, so by (5) in Definition 15, v is in $\text{Spl}(T, p)$. Thus, there is a splitting node w extending v in T before level m_p of T ; hence, there is a splitting node in T extending t . Therefore, T is perfect. \square

Fact 17. *Any strong coding tree codes a the universal triangle-free graph. Furthermore, given a strong coding tree T , any subtree of T which satisfies the Parallel 1's Criterion and is maximally splitting in T is also a strong coding tree.*

This fact may be easily checked by showing that such a tree satisfies the criterion $(A_3)^{\text{tree}}$. However, we shall be working with subtrees of a given tree \mathbb{T} (see Theorem 19) which are strongly similar to \mathbb{T} , and hence code \mathcal{H}_3 by virtue of the strong similarity between the subtree and the tree \mathbb{T} which codes \mathcal{H}_3 .

Definition 18. Finite strong triangle-free tree Any finite initial subtree of a strong coding tree is a *finite strong triangle-free tree*. These are exactly the finite trees which have all maximal nodes of the same length, and satisfy (2) - (8) of Definition 15.

We shall say that a strong coding tree T is *regular* if for each n , nodes in $\text{Spl}(T, n)$ extend to splitting nodes from lexicographic least to lexicographic larger. That is, for $n < \omega$ and $t, t' \in \text{Spl}(T, n)$, then $|\text{spl}(t)| > |\text{spl}(t')|$ if and only if $t <_{\text{lex}} t'$. Regularity is not necessary for the results, but can serve to make constructions concrete.

We now construct a strong coding tree \mathbb{T} . This should be thought of as a stretched and skewed version of the strong triangle-free tree \mathbb{S} which was constructed in Section 2. The passing numbers at the coding nodes in \mathbb{T} act exactly as the passing numbers of the coding nodes in \mathbb{S} ; hence they code \mathcal{H}_3 in the same order. By zipping the splitting nodes in intervals of \mathbb{T} between coding nodes in \mathbb{T} so that the splitting nodes between levels of c_n^T and c_{n+1}^T occur on the same level as c_{n+1}^T , we arrive back at a tree isomorphic to \mathbb{S} .

The strong coding tree \mathbb{T} which we will construct will be regular and will have the additional property that for each $n < \omega$, $c_{n+1}^T(l_n) = 1$. As in the case of \mathbb{S} , it will follow that no coding node in \mathbb{T} will split.

Theorem 19. *There is a strong coding tree \mathbb{T} .*

Proof. Let $\langle F_i : i < \omega \rangle$ be any sequence listing the finite subsets of ω such that each finite set appears cofinally often, and for each $i < \omega$, $\max(F_i) \leq i - 2$ and $F_{3i} = F_{3i+2} = \emptyset$. Let

$\langle u_i : i < \omega \rangle$ be an enumeration of all the nodes in $2^{<\omega}$ in such a way that each $|u_i| \leq i$. For each $n < \omega$, let c_n denote the n -th coding node in \mathbb{T} and let $l_n = |c_n|$. We construct a perfect coding tree $\mathbb{T} \subseteq 2^{<\omega}$ satisfying the following properties:

- (1) \mathbb{T} has no terminal nodes.
- (2) For $n = 4i + j$, $j < 3$, c_n satisfies requirement F_{3i+j} .
- (3) For $n = 4i + 3$, if u_i is in \mathbb{T} then c_n is a coding node extending u_i . If u_i is not in \mathbb{T} , then $c_n = 0^{l_{n-1}-1} \frown (1, 1) \frown 0^{q_n}$ where $q_n = l_n - (l_{n-1} + 1)$.
- (4) For each $n < \omega$, $c_n \frown 1 \notin \mathbb{T}$ and $c_{n+1}(l_n) = 1$.

To begin, define $\mathbb{T} \cap 2^0 = 2^0$, which is $\{\langle \rangle\}$, and define $\mathbb{T} \cap 2^1 = 2^1$. Let $\mathbb{T} \cap 2^2 = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle\}$. Define $c_0 = \langle 1, 0 \rangle$. Then $l_0 = 2$.

Note that $\mathbb{T} \cap 2^3$ is completely determined by $\mathbb{T} \cap 2^2$ and the definition of strong coding tree: Since $\langle 0 \rangle$ splits before level of c_0 , $\langle 0, 0 \rangle$ must be extended to $\langle 0, 0, 0 \rangle$ and $\langle 0, 1 \rangle$ must be extended to $\langle 0, 1, 1 \rangle$. Since $\langle 1, 0 \rangle$ does not split before level of c_0 , $\langle 1, 0 \rangle$ must be extended to $\langle 1, 0, 0 \rangle$. Thus, $\mathbb{T} \cap 2^3$ must equal $\{\langle 0, 0, 0 \rangle, \langle 0, 1, 1 \rangle, \langle 1, 0, 0 \rangle\}$. Note that $\mathbb{T} \cap 2^{\leq 3}$ is a finite regular, strongly skew, strong triangle-free tree.

We construct the next few levels up to the level of c_1 to illustrate the construction. $F_1 = \emptyset$. Since we will be satisfying (4), the next coding node c_1 will have to extend $\langle 0, 1, 1 \rangle$, since this is the only node in $\mathbb{T} \cap 2^3$ which has value 1 at l_0 . The knowledge that c_1 will extend $\langle 0, 1, 1 \rangle$ along with $\mathbb{T} \cap 2^3$ determines that $\text{Spl}(\mathbb{T}, 1) = \{\langle 0, 0, 0 \rangle, \langle 1, 0, 0 \rangle\}$, since these are the nodes in $\mathbb{T} \cap 2^{l_0+1}$ which have no parallel 1's with $\langle 0, 1, 1 \rangle = d_1 \upharpoonright 2^3$. Since we are making \mathbb{T} to be regular, $\langle 1, 0, 0 \rangle >_{\text{lex}} \langle 0, 0, 0 \rangle$ implies that $\langle 1, 0, 0 \rangle$ splits first. Let $\mathbb{T} \cap 2^4 = \{\langle 0, 0, 0, 0 \rangle, \langle 0, 1, 1, 0 \rangle, \langle 1, 0, 0, 0 \rangle, \langle 1, 0, 0, 1 \rangle\}$. Next, $\langle 0, 0, 0, 0 \rangle$ should split, since it is the only extension of $\langle 0, 0, 0 \rangle$ in $\mathbb{T} \cap 2^4$. Let

$$(3) \quad \mathbb{T} \cap 2^5 = \{\langle 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 0, 0, 1 \rangle, \langle 0, 1, 1, 0, 0 \rangle, \langle 1, 0, 0, 0, 0 \rangle, \langle 1, 0, 0, 1, 0 \rangle\}.$$

Let $c_1 = \langle 0, 1, 1, 0, 0 \rangle$. Notice that c_1 is the only extension of $\langle 0, 1, 1 \rangle$. Thus, $l_1 = 5$, $\text{spl}(\langle 1, 0, 0 \rangle) = \langle 1, 0, 0 \rangle$ and $\text{spl}(\langle 0, 0, 0 \rangle) = \langle 0, 0, 0, 0 \rangle$, and $\mathbb{T} \cap 2^5$ is a regular, strongly skew, strong triangle-free tree satisfying requirements F_0 and F_1 .

Note that $\mathbb{T} \cap 2^6$ is completely determined by the splitting between levels l_1 and l_2 in $\mathbb{T} \cap 2^5$.

$$(4) \quad \mathbb{T} \cap 2^6 = \{\langle 0, 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 0, 0, 1, 1 \rangle, \langle 0, 1, 1, 0, 0, 0 \rangle, \langle 1, 0, 0, 0, 0, 0 \rangle, \langle 1, 0, 0, 1, 0, 1 \rangle\}.$$

This constructs the tree \mathbb{T} up to level $l_1 + 1$ satisfying (1) - (4).

Suppose \mathbb{T} has been constructed up to the level of $l_{n-1} + 1$ satisfying (1) - (4). Enumerate the members of $\mathbb{T} \cap 2^{l_{n-1}+1}$ in decreasing lexicographical order as $\langle s_k : k < K \rangle$, where $K = |\mathbb{T} \cap 2^{l_{n-1}+1}|$. Let k_* denote the member of K such that s_{k_*} is to be extended to the n -th coding node c_n . We will show in each of the three cases how to choose k_* . Let K_S be the set of those $k < K$ such that for all $i < n$, $s_k(l_i) + s_{k_*}(l_i) \leq 1$. Thus, K_S is the set of indices of those nodes at level $l_{n-1} + 1$ which will split before the level of c_n ; $\text{Spl}(\mathbb{T}, n)$ equals $\{s_k : k \in K_S\}$. It follows that l_n will equal $l_{n-1} + |K_S| + 1$. Let $c_n = s_{k_*} \frown 0^{|K_S|}$.

There are three cases to consider regarding which $k < K$ should be k_* .

Case 1. $n = 4i$ or $n = 4i + 2$ for some $i < \omega$. Let n' denote $3i$ if $n = 4i$ and $3i + 2$ if $n = 4i + 2$. In this case, $F_{n'} = \emptyset$. Let $k_* = K - 2$. Since s_{K-1} is the lexicographic largest member of $\mathbb{T} \cap 2^{l_{n-1}+1}$, s_{K-1} must be $0^{l_{n-1}+1}$. Hence, s_{K-2} being lexicographic largest below

s_{K-1} , implies that $s_{K-2} = 0^{(l_{n-1}-1)} \frown (1, 1)$. Let $k_* = K - 2$. Note that any extension of s_{k_*} to a coding node will code an edge with c_{n-1} and with no other coding nodes c_i with $i < n - 1$.

Case 2. $n = 4i + 1$ for some $j < \omega$. If there is a pair $k < m$ of integers in F_{3i+1} such that $c_m(|c_k|) = 1$, then again take k_* to be $K - 2$. If for all pairs $k < m$ in F_{3i+1} , $c_m(|c_k|) = 0$, then since the $\mathbb{T} \cap 2^{\leq l_{n-1}+1}$ is strong triangle-free, there is at least one $k < K$ such that for each $j \leq n - 2$, $s_k(l_j) = 1$ if and only if $i \in F_{3i+1}$, and moreover, $s_k(l_{n-1}) = 1$. Choose one such k and label it k_* .

Case 3. $n = 4i + 3$ for some $i < \omega$. If $u_i \notin \mathbb{T} \cap 2^{\leq l_{n-1}}$, then let $k_* = K - 2$. If $u_i \in \mathbb{T} \cap 2^{\leq l_{n-1}}$, then let k_* be the index such that $s_{k_*} \supset u_i$, $s_{k_*}(l_{n-1}) = 1$, and $s_{k_*}(l_j) = 0$, for all $j < n - 1$ such that $|u_i| < l_j$.

Now, we show how to extend each s_k , $k \in K$, to level $l_n + 1$. For each $k \notin K_S$, extend s_k via all 0's to level $l_n + 1$. Enumerate K_S as $\langle k_i : i < |K_S| \rangle$, in increasing order; so $s_{k_i} <_{\text{lex}} s_{k_{i+1}}$ for each i . Let $\text{spl}(s_{k_i}) = s_{k_i} \frown 0^i$. Put into $\mathbb{T} \cap 2^{l_n+1}$ both $\text{spl}(s_{k_i}) \frown (1, 0, \dots, 0, 1)$ and $\text{spl}(s_{k_i}) \frown (0, \dots, 0)$. Precisely, letting $p = |K_S| - i$, $s_{k_i} \frown 0^{|K_S|}$ and $s_{k_i} \frown 0^i \frown 1 \frown 0^{p-2} \frown 1$ are the two extensions of s_{k_i} in $\mathbb{T} \cap 2^{l_n+1}$. Notice that for each $q < 2$, the $t \in \mathbb{T} \cap 2^{l_n+1}$ extending $\text{spl}(s_{k_i})^q$ has passing number $t(l_n) = q$. We have built \mathbb{T} to nodes of length 2^{l_n+1} . Technically, the nodes in \mathbb{T} constructed so far are the nodes at the splitting levels and the nodes at level l_n .

This completes the construction of \mathbb{T} , a prototypical regular, strongly skew, strong triangle-free tree in which the coding nodes code \mathcal{H}_3 . \square

Definition 20. Let T be any strong coding subtree of \mathbb{T} . For S a subtree of T , we write $S \leq T$ if and only if S is a strong coding tree and moreover S is strongly similar to T . Let $\mathcal{T}(T)$ denote the set of all strong coding trees S such that $S \leq T$.

Let $T \leq \mathbb{T}$. Note that each $S \in \mathcal{T}(T)$ codes a copy of \mathcal{H}_3 and is strongly similar to \mathbb{T} . The coding nodes of T , enumerated in order of increasing length, are denoted by $\langle c_i^T : i < \omega \rangle$. By a *critical node* of T , we mean either a coding node or a splitting node. Enumerate the set of critical nodes in T in order of increasing length as $\langle d_i^T : i < \omega \rangle$. Let $m_i = |d_i^T|$. The i -th level of T is defined to be

$$(5) \quad \text{Lev}_T(i) = \{t \in T : |t| = m_i\}.$$

Definition 21 (Finite Approximations). Let $T \leq \mathbb{T}$. For each $k < \omega$, define the k -th approximation to T to be $r_k(T) = \bigcup_{i < k} \text{Lev}_T(i)$. In particular, $r_0(T) = \emptyset$.

For $k < \omega$, a finite subtree $U \subseteq T$ is in $\mathcal{AT}_k(T)$ if and only if the following hold:

- (1) U is a finite strong triangle-free subtree of T , and
- (2) $U \stackrel{s}{\sim} r_k(T)$.

Notice each $U \in \mathcal{AT}_k(T)$ satisfies the Parallel 1's Criterion, by (2) and the fact that $r_k(T)$ does. Define

$$(6) \quad [U, T] = \{S \leq T : r_k(S) = U\}.$$

This is the set of all strong coding subtrees of T strongly similar to T and end-extending U .

4. A HALPERN-LAUCHLI-STYLE THEOREM FOR STRONG CODING TREES

This section begins the Ramsey theory content for strong coding trees, Theorem 22, which is an analogue of the strong tree version of the Halpern-Läuchli Theorem. This theorem forms

the basis for the Ramsey theorems in later sections regarding finite trees coding a fixed finite triangle-free graph. The following terminology will be used throughout.

Let T be a strong coding tree. Recall that \hat{T} denotes the set of all initial segments of members of T ; thus, for each $t \in T$, for all $m < |t|$, $t \upharpoonright m = t \cap 2^m \in \hat{T}$. Given a finite subtree A of T , let $\max(A)$ denote the set of all nodes in A which are maximal, that is terminal, in A . Let A^+ denote the set of *immediate successors* in \hat{T} of the members of $\max(A)$:

$$(7) \quad A^+ = \{s \frown i : s \in \max(A), i < 2, \text{ and } s \frown i \in \hat{T}\}.$$

A set $X \subseteq T$ is a *level set* if all nodes in X have the same length. Let A be a finite subtree of T . Let A_e be a subset of A^+ , and let C be a subtree of T such that $C \setminus A$ is a level set, each node in $C \setminus A$ extends a unique node in A_e and each node in A_e is extended by a unique node in C . $\text{Ext}_T(A, C)$ is the collection of all $X \subseteq T$ such that the members of X extend the members of A_e , and $A \cup X \overset{s}{\sim} C$. In particular, if $A \cup X = r_{n+1}(T)$, then it follows that $A = r_n(T)$ and $\text{Ext}_T(A, C) = r_{n+1}[A, T]$. Given finite trees $A \subseteq X$, and given $A_e \subseteq A^+$, we will say that X *extends* A_e if each node in $X \setminus A$ extends a unique member of A_e , and each member of A_e is extended by a unique member of X .

We prove a general theorem which encompasses colorings of two different types of level set extensions of a fixed finite tree: The extension either contains a splitting node (Case (a)) or a coding node (Case (b)). The proof uses ideas from Harrington's forcing proof of the Halpern-Läuchli Theorem though new considerations arise that have to be dealt with in the setting of triangle-free trees. The main reasons that new forcings are needed are that there are two types of nodes, coding and splitting nodes, and we want to force in such a way that the extensions achieving homogeneity will be robust enough that they can be extended to strong coding trees contained within the original strong coding tree. The former is responsible for there being Cases (a) and (b), and the latter is responsible for the strong definition of the partial ordering on the forcing. The forcings will consist of certain finite subtrees of a given tree T , but the partial ordering will be stronger than the partial ordering of subtree, so these are not simply Cohen forcings.

Remark. Although the proof uses the set-theoretic technique of forcing, the whole construction takes place in the original model of ZFC, not some generic extension. The forcing should be thought of as conducting an unbounded search for a finite object, namely the finite set of nodes of a prescribed form where homogeneity is attained. Thus, the result and its proof hold using only the standard axioms of mathematics, and as such, it should be possible to find a non-forcing proof.

We provide the set-up for the two cases before stating the theorem.

The Set-up for Theorem 22. Let T be any strong coding tree. For each $s \in T$, if $i \in \{0, 1\}$ and $s \frown i$ is in \hat{T} , then we say that $s \frown i$ is an *immediate extension* of s . We will also call i an immediate extension of s . We say that a subset X of T is a *level set* if all members of X have the same length.

Let A be a finite subtree of T satisfying the Parallel 1's Criterion. A may have maximal nodes at different levels. Let A_* be a non-empty subset of the maximal nodes in A such that A_* is a level set and all nodes in A_* have the maximal length in A . Let A_e be the set of all immediate successors of the members of A_* . Suppose C is a subtree of T containing A such

that C satisfies the Parallel 1's Criterion, each node in A_e is extended by a node in C , and each node in $C \setminus A$ extends exactly one node in A_e . The two cases considered in the next theorem are the following:

Case (a) $C \setminus A$ is a level set and $C \setminus A$ contains a splitting node.

Case (b) $C \setminus A$ is a level set and $C \setminus A$ contains a coding node. Recall that the existence of a strong similarity h between $A \cup X$ and C implies that for each $s \in X^+$, the immediate extension of s is the same as the immediate extension of $h(s)$. Thus, passing numbers are preserved in this case.

Note that in both Cases (a) and (b), $\text{Ext}_T(A, C)$ is the collection of those $X \subseteq T$ such that the members of X end-extend the members of A_e and $A \cup X \stackrel{s}{\sim} C$.

Recall from Definition 21 that $[B, T]$ denotes the set of all strong coding trees $S \leq T$ such that S end-extends B . For any finite strong triangle-free tree or strong coding tree U , we write

$$(8) \quad B \sqsubset U$$

to denote that U end-extends B ; that is, for some k , $r_k(U) = B$. Given $B \in \mathcal{AT}_k(T)$ and $m > k$, let

$$(9) \quad r_m[B, T]$$

denote the set of all $U \in \mathcal{AT}_m(T)$ such that $r_k(U) = B$. For any $l < \omega$ and any tree $U \subseteq \mathbb{T}$, let

$$(10) \quad U \upharpoonright l = \{s \in U : |s| \leq l\} \cup \{t \upharpoonright l : t \in U \text{ and } |t| > l\}.$$

In Case (b), given A and C as above, for each $X \in \text{Ext}_T(A, C)$, let $\text{Ext}_T(A, C; X)$ denote the set of $Y \in \text{Ext}_T(A, C)$ such that Y extends X .

Theorem 22. *Let $T \subseteq \mathbb{T}$ be any strong coding tree, and let A, A_e, C satisfy Case (a) or Case (b) above, with both A and C satisfying the Parallel 1's Criterion. Let k be minimal such that $A \subseteq r_k(T)$, and let B denote $r_k(T)$.*

In Case (a), given any coloring $c : \text{Ext}_T(A, C) \rightarrow 2$, there is a strong coding tree $S \in [B, T]$ such that c is monochromatic on $\text{Ext}_S(A, C)$.

In Case (b), given any fixed $X \in \text{Ext}_T(A, C)$ and a coloring $c : \text{Ext}_T(A, C)$, letting k be minimal such that $r_{k+1}(T)$ extends X , there is a strong coding tree $S \in [r_k(T), T]$ such that c is monochromatic on $\text{Ext}_S(A, C; X)$.

Remark. In terms of topological Ramsey space theory, the previous theorem implies the pigeonhole principle **Axiom A.4** holds, in Todorćević's axiom system guaranteeing a topological Ramsey space. (See Chapter 5 in [12].)

Lemma 23 (Passing Number Choice Extension Lemma). *Let T be a strong coding tree and B be any finite subtree of T such that $\max(B)$ is a level set. Let l_B denote the length of the members of B and list the nodes of B^+ as s_0, \dots, s_n . Let j be such that c_j^T is the least coding node in T of length greater than or equal to l_B . For each $i \leq n$, let t_i be the leftmost extension of s_i of length $|c_j^T| + 1$. Fix any $d \leq n$. For each $i \neq d$, if t_i and t_d have no parallel 1's, fix any $m_i \in \{0, 1\}$; if t_i and t_d have parallel 1's, let $m_i = 0$. In particular, $m_d = 0$.*

Then for each $j' \geq j + 2$, there is a coding node $c_{k'}^T$ with $k \geq j'$ extending t_d , and there are extensions $u_i \supseteq t_i$ of length $|c_{k'}^T| + 1$ such that $u_i(|c_{k'}^T|) = m_i$ for each $i \leq n$. We set $u_d = c_{k'}^T$.

Furthermore, the u_i can be chosen so that no new parallel 1's are added below the level of c_k^T ; that is, u_i and $u_{i'}$ have parallel 1's if and only if $t_i \upharpoonright |c_k^T|$ and $t_{i'} \upharpoonright |c_k^T|$ have parallel 1's. In particular, if $B \cup \{t_i : i \leq n\}$ satisfies the Parallel 1's Criterion, then $B \cup \{u_i : i \leq n\}$ also satisfies the Parallel 1's Criterion.

Proof. Assume the hypotheses of the lemma and let $j' \geq j+2$ be given. Take $k \geq j'$ minimal such that $c_k^T \supseteq t_d$, and let $u_d = c_k^T$. For each $i \leq n$, $i \neq d$, extend t_i via its leftmost extension to the level of $|c_{k-1}^T| + 1$, and label it t'_i . Thus, t'_i is the leftmost extension of s_i to level $|c_{k-1}^T| + 1$, so $t'_i(|c_{k-1}^T|) = 0$. Notice that for each $i \neq i'$, t'_i has a parallel 1 with $t'_{i'}$ if and only if t_i has a parallel 1 with $t_{i'}$.

For each $i \neq d$ for which either t_i and t_d have a parallel 1 or else $m_i = 0$, let u_i be the leftmost extension of t'_i of length $|c_k^T| + 1$. Then $u_i(|c_k^T|) = 0 = m_i$. Moreover, for each $i' \neq i$, whenever u_i and $u_{i'}$ have parallel 1's, so do t_i and $t_{i'}$. That is, if there is an $l \in (|c_j^T|, |c_k^T|]$ such that $u_i(l) = u_{i'}(l) = 1$, then there is an $l' \leq |c_j^T|$ such that $u_i(l') = u_{i'}(l') = 1$. Suppose now that $i \neq d$, s_i and s_d have no parallel 1's, and $m_i = 1$. Let u_i be the rightmost extension of t'_i to length $|c_k^T| + 1$. Then $u_i(|c_k^T|) = 1 = m_i$.

If u_i and $u_{i'}$ have parallel 1's below the level of $|s_*|$, then t_i and $t_{i'}$ must have parallel 1's. If the least occurrence of parallel 1's between u_i and $u_{i'}$ is above the level of $|s_*|$, then this is witnessed by the coding node c_k^T . Hence, if $B \cup \{t_i : i \leq n\}$ satisfies the Parallel 1's Criterion, then $B \cup \{u_i : i \leq n\}$ also satisfies the Parallel 1's Criterion. \square

The following fact will be used throughout the proof of the theorem.

Fact 24. Suppose T is a strong coding tree, c is a coding node in T , and v and w are nodes in T of length $|c| + 1$ with no parallel 1's. Then for any $l > |c|$, extending v by its leftmost extensions in T to length l produces two nodes with no parallel 1's with any extension of w to length l .

The following terminology will help streamline the proof of the theorem.

Definition 25. If l is the length of a coding node d and $j \in \{0, 1\}$, we say that u has passing number 1 at d if the length of u is greater than l and $u(l) = j$. We say that two nodes u, v of T have parallel 1's at the level l if $u(l) = v(l) = 1$. u and v have first parallel 1's at level l if l is least such that $u(l) = v(l) = 1$. A coding node d is the least witness of parallel 1's for u and v if d is the coding node of least length such that u and v both have passing number 1 at d .

Proof of Theorem 22. Let T, A, A_e, C be given satisfying the hypotheses of Case (a) or (b). Let $d+1$ equal the number of nodes in A_e . List the nodes of A_e as s_0, \dots, s_d , letting s_d denote the node of A_e that will be extended to a splitting node in Case (a) and to a coding node in Case (b) in order to extend A_e to C , and more generally, to extend A to a member of $\text{Ext}_T(A, C)$. In Case (b), letting t_d denote the coding node in $\max(C)$ (hence extending s_d), and letting t_i^+ denote the member in $\max(C)^+$ extending s_i , let I_0 denote the set of all $i < d$ such that $t_i^+(|t_d|) = 0$ and let I_1 denote the set of all $i < d$ such that $t_i^+(|t_d|) = 1$.

Without loss of generality, and with no infringement on the results or its proof, we may assume that A_e includes a member of $0^{<\omega}$ and that, in any $X \in \text{Ext}_T(A, C)$, this member is to be extended to another member of $0^{<\omega}$. This is so as to uniformize the presentation, though it is not necessary for the proof. Let i_0 denote the integer such that s_{i_0} is the node of A_e which is all 0's and has the same length as the other nodes in A_e . Notice that i_0 can

equal d only if we are in Case (a) and moreover the leftmost maximal node of C is a splitting node in T .

Let L denote the collection of all $l < \omega$ such that there is a member of $\text{Ext}_T(A, C)$ with maximal nodes of length l . In Case (a), L is exactly the set of all $l < \omega$ for which there is a splitting node of length l extending s_d , and in Case (b), L is exactly the set of all $l < \omega$ for which there is a coding node of length l extending s_d . For each $i \in (d+1) \setminus \{i_0\}$, let $T_i = \{t \in T : t \supseteq s_i\}$. Let $T_{i_0} = \{t \in T \cap 0^{<\omega} : t \supseteq s_{i_0}\}$, the collection of all leftmost nodes in T extending s_{i_0} .

Let $\kappa = \beth_{2d}$. The following forcing notion \mathbb{P} will add κ many paths through each T_i , $i \in d \setminus \{i_0\}$; one path through T_{i_0} (though allowing κ many ordinals labeling this path in order to simplify notation); and one path through T_d . Both Cases are handled quite similarly, so we present a general proof, indicating the steps at which the two cases require different approaches.

Case (a). \mathbb{P} is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \rightarrow T \upharpoonright l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

- (i) $p(d)$ is *the* splitting node extending s_d at level l_p ;
- (ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$.

Case (b). \mathbb{P} is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \rightarrow T \upharpoonright l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

- (i) $p(d)$ is *the* coding node extending s_d at level l_p ;
- (ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$.
- (iii) For each $\delta \in \vec{\delta}_p$, $j \in \{0, 1\}$, and $i \in I_j$, the immediate extension of $p(i, \delta)$ in T is j .

Remark. \mathbb{P} will be adding one path through T_{i_0} with κ many labels, since $T_{i_0} \subseteq 0^{<\omega}$. This does not hurt the arguments and serves to simplify notation.

In both Cases (a) and (b), the partial ordering on \mathbb{P} is defined as follows: $q \leq p$ if and only if either

- (1) $l_q = l_p$ and $q \supseteq p$ (so also $\vec{\delta}_q \supseteq \vec{\delta}_p$); or else
- (2) $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and
 - (i) $q(d) \supset p(d)$, and for each $\delta \in \vec{\delta}_p$ and $i < d$, $q(i, \delta) \supset p(i, \delta)$;
 - (ii) Whenever $(\alpha_0, \dots, \alpha_{d-1})$ is a strictly increasing sequence in $(\vec{\delta}_p)^d$ and $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\} \in \text{Ext}_T(A, C)$, then also $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\} \in \text{Ext}_T(A, C)$.

The proof of the theorem proceeds in three sections. The first section proves that \mathbb{P} is an atomless partial order. The second section proves Lemma 27 which is then applied repeatedly in the third section to build the tree $S \in [B, T]$ which is homogeneous on $\text{Ext}_S(A, C)$ in Case (a), and on $\text{Ext}_S(A, C; X)$ in Case (b).

Section 1. \mathbb{P} is an atomless partial ordering.

Claim 1. (\mathbb{P}, \leq) is a partial ordering.

Proof. The order \leq on \mathbb{P} is clearly reflexive and symmetric. Transitivity follows from the fact that the requirement (2)(ii) is a transitive property: If $\vec{\delta}_r \supseteq \vec{\delta}_q \supseteq \vec{\delta}_p$ and $r \leq q$ and $q \leq p$ are both witnessed by satisfying (2), then whenever $(\alpha_0, \dots, \alpha_{d-1}) \in \vec{\delta}_p$ and $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\} \in \text{Ext}_T(A, C)$, then also $\{q(i, \alpha_i) : i < d\} \cup \{p(d)\} \in \text{Ext}_T(A, C)$ and hence $\{r(i, \alpha_i) : i < d\} \cup \{p(d)\} \in \text{Ext}_T(A, C)$, by (2) in the definition of the partial ordering on \mathbb{P} . In the case that either one or both of $r \leq q$ and $q \leq p$ are witnessed by satisfying (1) in the definition of \leq , transitivity is trivial. \square

Next, we show that \mathbb{P} is atomless by proving the following stronger claim that the set of all conditions in \mathbb{P} in which the minimal length of any node in the condition is longer than a given number is dense in \mathbb{P} .

Claim 2. For each $p \in \mathbb{P}$ and each $n > l_p$, there is a $q \in \mathbb{P}$ with $l_q > n$ satisfying $q < p$.

Proof. Let $p \in \mathbb{P}$ and $n > l_p$. Let $\vec{\delta}_q = \vec{\delta}_p$. In Case (a), take $q(d)$ to be a splitting node in T extending $p(d)$ such that $|q(d)| > n$. For each $(i, \delta) \in d \times \vec{\delta}_p$, simply let $q(i, \delta)$ be the leftmost extension of $p(i, \delta)$ in T of length l_q .

In Case (b), let $q(d)$ be a coding node in T extending $p(d)$ of length greater than the maximum of n and the least coding node in T above $p(d)$. By the Passing Number Choice Lemma 23, there are $q(i, \delta)$, $i \leq d$ and $\delta \in \vec{\delta}_p$, such that $q(i, \delta)$ extends $p(i, \delta)$ to the level of $q(d)$ and the following hold: For each $j \in \{0, 1\}$, $i \in I_j$ if and only if the immediate extension of $q(i, \delta)$ is j . That is, the passing number $q(i, \delta)$ at $q(d)$ is the same as the passing number of $p(i, \delta)$ at $p(d)$. Moreover, it follows from the construction in the Passing Number Choice Lemma 23 that for each increasing sequence $(\alpha_i : i < d)$ in $\vec{\delta}_p$ for which $A \cup \{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ satisfies the Parallel 1's Criterion, also $A \cup \{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ satisfies the Parallel 1's Criterion. Therefore, for each increasing sequence $(\alpha_i : i < d)$ in $\vec{\delta}_p$ for which $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ is a member of $\text{Ext}_T(A, C)$, $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ is also a member of $\text{Ext}_T(A, C)$. Thus, $q \in \mathbb{P}$ and $q < p$. \square

Whenever the steps for Cases (a) and (b) are the same, we will refer to the splitting node in Case (a) and the coding node in Case (b) simply as the *critical node*, if no ambiguity will arise by doing so.

Let \dot{b}_d be a \mathbb{P} -name for the generic path through T_d ; that is, $\dot{b}_d = \{\langle p(d), p \rangle : p \in \mathbb{P}\}$. Note that for each $p \in \mathbb{P}$, p forces that $\dot{b}_d \restriction l_p = p(d)$. By Claim 2, it is dense to force a critical node in \dot{b}_d above any given level in T , so $\mathbf{1}_{\mathbb{P}}$ forces that the set of levels of critical nodes in \dot{b}_d is cofinal in ω . Thus, given any generic G filter for \mathbb{P} , $\dot{b}_d^G = \{p(d) : p \in G\}$ is a cofinal path of critical nodes in T_d . Let \dot{L}_d be a \mathbb{P} -name for the set of lengths of critical nodes in \dot{b}_d . Note that $\mathbf{1}_{\mathbb{P}} \Vdash \dot{L}_d \subseteq L$. Let $\dot{\mathcal{U}}$ be a \mathbb{P} -name for a non-principal ultrafilter on \dot{L}_d .

For each $i < d$ and $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ be a \mathbb{P} -name for the α -th generic branch through T_i ; that is, $\dot{b}_{i,\alpha} = \{\langle p(i, \alpha), p \rangle : p \in \mathbb{P} \text{ and } \alpha \in \vec{\delta}_p\}$. For $i < d$ in Cases (a) and (b), for any condition $p \in \mathbb{P}$ and $\alpha \in \vec{\delta}_p$, p forces that $\dot{b}_{i,\alpha} \restriction l_p = p(i, \alpha)$.

For ease of notation, we shall write sets $\{\alpha_i : i < d\}$ in $[\kappa]^d$ as vectors $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle$ in strictly increasing order. For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle \in [\kappa]^d$, rather than writing out

$$(11) \quad \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$$

each time we wish to refer to these generic branches, we shall simply

$$(12) \quad \text{let } \dot{b}_{\vec{\alpha}} \text{ denote } \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle,$$

since the branch \dot{b}_d being unique causes no ambiguity. For any $l < \omega$,

$$(13) \quad \text{let } \dot{b}_{\vec{\alpha}} \restriction l \text{ denote } \{\dot{b}_{i,\alpha_i} \restriction l : i < d\} \cup \{\dot{b}_d \restriction l\}.$$

Let $c : \text{Ext}_T(A, C) \rightarrow 2$ be given coloring. In both Cases, using the abbreviations just defined, c is a coloring on sets of nodes of the form $\dot{b}_{\vec{\alpha}} \restriction l$ whenever this is forced to be a member of $\text{Ext}_T(A, C)$.

Section 2. The goal of this section is to prove Claims 3 and 4 and Lemma 27. In summary, they secure that there are infinite pairwise disjoint sets $K'_i \subseteq \kappa$, $i < d$, and a set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i < d} K'_i\}$ which are compatible, have the same images in T , and such that for some fixed ε^* , for each $\vec{\alpha} \in \prod_{i < d} K'_i$, $p_{\vec{\alpha}}$ forces $c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon^*$ for ultrafilter many $l \in \dot{L}_d$. Moreover, there are t_i^* , $i \leq d$, such that for each $\vec{\alpha} \in \prod_{i < d} K'_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$. This lemma serves as the foundation for the fusion process obtaining S in Section 3 of the proof. In this section, there are no differences between the arguments for Cases (a) and (b).

For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

- (1) $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$.
- (2) $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p_{\vec{\alpha}}(d)\} \in \text{Ext}_T(A, C)$.
- (3) $p_{\vec{\alpha}} \Vdash$ “There is an $\varepsilon \in 2$ such that $c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon$ for $\dot{\mathcal{U}}$ many l in \dot{L}_d .”
- (4) $p_{\vec{\alpha}}$ decides a value for ε , call it $\varepsilon_{\vec{\alpha}}$.
- (5) $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p_{\vec{\alpha}}(d)\}) = \varepsilon_{\vec{\alpha}}$.

Properties (1) and (2) can be guaranteed as follows: For each $i \leq d$, let c_i denote the member of C such that $c_i \supseteq s_i$. For each $\vec{\alpha} \in [\kappa]^d$, let

$$p_{\vec{\alpha}}^0 = \{\langle (i, \delta), c_i \rangle : i < d, \delta \in \vec{\alpha}\} \cup \{\langle d, c_d \rangle\}.$$

Then $p_{\vec{\alpha}}^0$ is a condition in \mathbb{P} , $\vec{\delta}_{p_{\vec{\alpha}}^0} = \vec{\alpha}$, and $\{p_{\vec{\alpha}}^0(i, \alpha_i) : i < d\} \cup \{p_{\vec{\alpha}}^0(d)\}$ is a member of $\text{Ext}_T(A, C)$. It follows by the definition of the partial ordering on \mathbb{P} that any extension $p \leq p_{\vec{\alpha}}^0$ must have $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ also being a member of $\text{Ext}_T(A, C)$.

By Claim 2, $1_{\mathbb{P}}$ forces that \dot{L}_d is infinite. $\dot{\mathcal{U}}$ is a name for an ultrafilter on \dot{L}_d . Thus, there is an extension $p_{\vec{\alpha}}^1 \leq p_{\vec{\alpha}}^0$ forcing $c(\dot{b}_{\vec{\alpha}} \restriction l)$ to be the same for $\dot{\mathcal{U}}$ many $l \in \dot{L}_d$, giving (3). For Property (4), since \mathbb{P} is a forcing notion, there is a $p_{\vec{\alpha}}^2 \leq p_{\vec{\alpha}}^1$ deciding a value $\varepsilon_{\vec{\alpha}}$ for which $p_{\vec{\alpha}}^2$ forces that $c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l in \dot{L}_d .

By extending $p_{\vec{\alpha}}^2$ if necessary, we may take $p_{\vec{\alpha}}^3 \leq p_{\vec{\alpha}}^2$ such that $p_{\vec{\alpha}}^3$ decides $c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$, for some $l \in \dot{L}$ such that l is greater than or equal to the lengths of the nodes in $p_{\vec{\alpha}}^2$, and such that $l \leq l_{p_{\vec{\alpha}}^3}$. Let $p_{\vec{\alpha}}$ be $p_{\vec{\alpha}}^3$ truncated to level l . Then $p_{\vec{\alpha}} \leq p_{\vec{\alpha}}^2$ so it still satisfies (1) through (4). Since $\{p_{\vec{\alpha}}(i, \alpha_i)\} \cup \{p_{\vec{\alpha}}(d)\}$ is what $p_{\vec{\alpha}}$ forces $\dot{b}_{\vec{\alpha}} \restriction l$ to be, it follows that $p_{\vec{\alpha}}$ forces $c(\{p_{\vec{\alpha}}(i, \alpha_i)\} \cup \{p_{\vec{\alpha}}(d)\}) = \varepsilon_{\vec{\alpha}}$, so (5) holds.

Recall the following theorem of Erdős and Rado from [1].

Theorem 26 (Erdős-Rado). *For $r < \omega$ and μ an infinite cardinal,*

$$\beth_r(\mu)^+ \rightarrow (\mu^+)_{\mu}^{r+1}.$$

We are assuming $\kappa = \beth_{2d}$, which is at least $\beth_{2d-1}(\aleph_0)^+$, so $\kappa \rightarrow (\aleph_1)_{\aleph_0}^{2d}$ by Theorem 26.

Now we prepare for an application of the Erdős-Rado Theorem. Given two sets of ordinals J, K we shall write $J < K$ if and only if every member of J is less than every member of K . Let $D_e = \{0, 2, \dots, 2d-2\}$ and $D_o = \{1, 3, \dots, 2d-1\}$, the sets of even and odd integers less than $2d$, respectively. Let \mathcal{I} denote the collection of all functions $\iota : 2d \rightarrow 2d$ such that $\iota \upharpoonright D_e$ and $\iota \upharpoonright D_o$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \dots < \{\iota(2d-2), \iota(2d-1)\}$. Thus, each ι codes two strictly increasing sequences $\iota \upharpoonright D_e$ and $\iota \upharpoonright D_o$, each of length d . For $\vec{\theta} \in [\kappa]^{2d}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $(\theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d-2)}), (\theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d-1)})$, both of which are members of $[\kappa]^d$. Denote these as $\iota_e(\vec{\theta})$ and $\iota_o(\vec{\theta})$, respectively. To ease notation, let $\vec{\delta}_{\vec{\alpha}}$ denote $\vec{\delta}_{p_{\vec{\alpha}}}$, $k_{\vec{\alpha}}$ denote $|\vec{\delta}_{\vec{\alpha}}|$, and let $l_{\vec{\alpha}}$ denote $l_{p_{\vec{\alpha}}}$. Let $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$ denote the enumeration of $\vec{\delta}_{\vec{\alpha}}$ in increasing order.

Define a coloring f on $[\kappa]^{2d}$ into countably many colors as follows: Given $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, to reduce the number of subscripts, letting $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, define

$$(14) \quad f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p_{\vec{\alpha}}(d), \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \\ \langle \langle i, j \rangle : i < d, j < k_{\vec{\alpha}}, \text{ and } \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \langle \langle j, k \rangle : j < k_{\vec{\alpha}}, k < k_{\vec{\beta}}, \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle.$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where \mathcal{I} is given some fixed ordering. Since the range of f is countable, applying the Erdős-Rado Theorem, we obtain a subset $K \subseteq \kappa$ of cardinality \aleph_1 which is homogeneous for f . Take $K' \subseteq K$ such that between each two members of K' there is a member of K and $\min(K') > \min(K)$. Take subsets $K'_i \subseteq K'$ such that $K'_0 < \dots < K'_{d-1}$ and each $|K'_i| = \aleph_0$.

Claim 3. *There are $\varepsilon^* \in 2$, $k^* \in \omega$, t_d , and $\langle t_{i,j} : j < k^* \rangle$, $i < d$, such that $\varepsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, $p_{\vec{\alpha}}(d) = t_d$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle$, for all $\vec{\alpha} \in \prod_{i < d} K'_i$ and each $i \leq d$.*

Proof. Let ι be the member in \mathcal{I} which is the identity function on $2d$. For any pair $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$, there are $\vec{\theta}, \vec{\theta}' \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\beta} = \iota_e(\vec{\theta}')$. Since $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta}')$, it follows that $\varepsilon_{\vec{\alpha}} = \varepsilon_{\vec{\beta}}$, $k_{\vec{\alpha}} = k_{\vec{\beta}}$, $p_{\vec{\alpha}}(d) = p_{\vec{\beta}}(d)$, and $\langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle = \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle$. \square

Let l^* denote the height of the node t_d . Since each $p_{\vec{\alpha}}$ is a condition, l^* must be in L . Note that for both Cases (a) and (b), all the nodes t_d and $t_{i,j}$, $j < k^*$, $i < d$, are all of the same length l^* .

Claim 4. *Given any $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.*

Proof. Let $\vec{\alpha}, \vec{\beta}$ be members of $\prod_{i < d} K'_i$ and suppose that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ for some $j, j' < k^*$. For each $i < d$, let ρ_i be the relation from among $\{<, =, >\}$ such that $\alpha_i \rho_i \beta_i$. Let ι be the member of \mathcal{I} such that for each $\vec{\gamma} \in [K]^d$ and each $i < d$, $\theta_{\iota(2i)} \rho_i \theta_{\iota(2i+1)}$. Then there is a $\vec{\theta} \in [K]^{2d}$ such that $\iota_e(\vec{\theta}) = \vec{\alpha}$ and $\iota_o(\vec{\theta}) = \vec{\beta}$. Since between any two members of K' there is a member of K , there is a $\vec{\gamma} \in [K]^d$ such that for each $i < d$, $\alpha_i \rho_i \gamma_i$ and $\gamma_i \rho_i \beta_i$, and furthermore, for each $i < d-1$, $\{\alpha_i, \beta_i < \gamma_i\} < \{\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}\}$. Given that $\alpha_i \rho_i \gamma_i$ and $\gamma_i \rho_i \beta_i$ for each $i < d$, there are $\vec{\mu}, \vec{\nu} \in [K]^{2d}$ such that $\iota_e(\vec{\mu}) = \vec{\alpha}$, $\iota_o(\vec{\mu}) = \vec{\gamma}$, $\iota_e(\vec{\nu}) = \vec{\gamma}$, and $\iota_o(\vec{\nu}) = \vec{\beta}$. Since $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, the pair $\langle j, j' \rangle$ is in the last sequence in $f(\iota, \vec{\theta})$. Since

$f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$, also $\langle j, j' \rangle$ is in the last sequence in $f(\iota, \vec{\mu})$ and $f(\iota, \vec{\nu})$. It follows that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(j')$ and $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(j')$. Hence, $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\gamma}}(j')$, and therefore j must equal j' . \square

For any $\vec{\alpha} \in \prod_{i < d} K'_i$ and any $\iota \in \mathcal{I}$, there is a $\vec{\theta} \in [K]^{2d}$ such that $\vec{\alpha} = \iota_o(\vec{\theta})$. By homogeneity of f and by the first sequence in the second line of equation (14), there is a strictly increasing sequence $\langle j_i : i < d \rangle$ of members of k^* such that for each $\vec{\alpha} \in \prod_{i < d} K'_i$, $\vec{\delta}_{\vec{\alpha}}(j_i) = \alpha_i$. For each $i < d$, let t_i^* denote t_{i, j_i} . Then for each $i < d$ and each $\vec{\alpha} \in \prod_{i < d} K'_i$,

$$(15) \quad p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \vec{\delta}_{\vec{\alpha}}(j_i)) = t_{i, j_i} = t_i^*.$$

Let t_d^* denote t_d .

Lemma 27. *The set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i < d} K'_i\}$ is compatible.*

There are infinite pairwise disjoint sets $K'_i \subseteq \kappa$, $i < d$, and a set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i < d} K'_i\}$ which are compatible, have the same images in T , and such that for some fixed ε^ , for each $\vec{\alpha} \in \prod_{i < d} K'_i$, $p_{\vec{\alpha}}$ forces $c(\dot{b}_{\vec{\alpha}} \upharpoonright l) = \varepsilon^*$ for ultrafilter many $l \in \dot{L}_d$. Moreover, there are t_i^* , $i \leq d$, such that for each $\vec{\alpha} \in \prod_{i < d} K'_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$. This lemma serves as the foundation for the fusion process obtaining S in Section 3 of the proof.*

Proof. Suppose toward a contradiction that there are $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$ such that $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ are incompatible. By Claim 3, for each $i < d$ and $j < k^*$,

$$(16) \quad p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i, j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)).$$

Thus, the only way $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ can be incompatible is if there are $i < d$ and $j, j' < k^*$ such that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ but $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) \neq p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j'))$. Since $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i, j}$ and $p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j')) = t_{i, j'}$, this would imply $j \neq j'$. But by Claim 4, $j \neq j'$ implies that $\delta_{\vec{\alpha}}(j) \neq \delta_{\vec{\beta}}(j')$, a contradiction. Therefore, $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ must be compatible. \square

Section 3. In this last section of the proof, for Case (a) we build a member $S \in [B, T]$ such that $\text{Ext}_S(A, C)$ is homogeneous for c , with color ε^* . For Case (b) given any fixed $X \in \text{Ext}_T(A, C)$, we build a member $S \in [r_k(T), T]$ such that $\text{Ext}_S(A, C; X)$ is homogeneous for c with color ε^* , where $k + 1$ is least such that $X \subseteq r_{k+1}(T)$. Cases (a) and (b) are now handled separately.

Case (a). Let $M = \{m_j : j < \omega\}$ be the strictly increasing enumeration of those $m > k$ such that the maximal splitting node in $r_m(T)$ extends s_d . Note that M is exactly the set of those $m > k$ for which there is a member X of $\text{Ext}_T(A, C)$ such that the maximal critical node of X is the same as the maximal critical node of $r_m(T)$. Further note that M is the collection of exactly those m for which the maximal length of the nodes $r_m(T)$ is in L . We will find $U_m \in r_m[B, T]$, $m \in M$, such that $B \sqsubset U_{m_0} \sqsubset U_{m_1} \sqsubset \dots$ and c takes color ε^* on $\text{Ext}_{U_{m_j}}(A, C)$. Then letting $S = \bigcup_{j < \omega} U_{m_j}$ will yield S to be a member of $[B, T]$ for which $\text{Ext}_S(A, C)$ is homogeneous for c , with color ε^* .

First extend each node in B^+ to level l^* as follows. The members of A_e are extended to the nodes $\{t_i^* : i \leq d\}$. Extend each node u in $B^+ \setminus A_e$ to its leftmost extension in $T \upharpoonright l^*$ and label that extension u^* . The set

$$(17) \quad U^* = \{t_i^* : i \leq d\} \cup \{u^* : u \in B^+ \setminus A_e\}$$

extends each member of B^+ to a unique node such that any parallel 1's between members of U^* are already witnessed in B^+ , so U^* also satisfies the Parallel 1's Criterion. So we are free to extend the members of U^* to build an $S \in [B, T]$. The members of U^* will not be a level of S , but the members of S above B will all extend the members of U^* .

If $m_0 > k + 1$, extend above U^* into T to construct a member $U_{m_0-1} \in r_{m_0-1}[B, T]$. Otherwise, $m_0 = k + 1$ and both B and U^* are members of \mathcal{AR}_{m_0-1} ; in this case, let $U_{m_0-1} = U^*$. Note that $\text{Ext}_{U_{m_0-1}}(A, C) = \emptyset$.

Induction Assumption: Assume $j < \omega$ and we have constructed U_{m_j-1} so that every member of $\text{Ext}_{U_{m_j-1}}(A, C)$ is colored ε^* by c .

Fix some $Y_j \in r_{m_j}[U_{m_j-1}, T]$ and let V_j denote $\max(Y_j)$. Let n_j denote the length of the members of V_j . Note that n_j is a member of L . The nodes in V_j will not be in the tree S we are constructing; rather, we will extend the nodes in V_j to construct $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$.

Let $q(d)$ be the splitting node in V_j . For each $i < d$ such that s_i and s_d do not have parallel 1's, let X_i be the set of nodes u in $V_j \cap T_i$ such that u and $q(d)$ do not have any parallel 1's. Notice that each node in X_i is in some member of $\text{Ext}_T(A, C)$, since extending all the other t_i^* via their leftmost extensions along with $q(d)$ constructs a member of $\text{Ext}_T(A, C)$. For each $i < d$ such that s_i and s_d have parallel 1's, let $X_i = \max(V_j) \cap T_i$. In this case, all extensions of t_i^* may be nodes in some member of $\text{Ext}_T(A, C)$, since any parallel 1's with an extension of t_d^* are already witnessed by A^+ .

For each $i < d$, let J_i be a subset of K'_i with the same size as X_i . For each $i < d$, label the nodes in X_i as $\{q(i, \delta) : \delta \in J_i\}$. Let \vec{J} denote the set of those $\langle \alpha_0, \dots, \alpha_{d-1} \rangle \in \prod_{i < d} J_i$ such that the set $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ is in $\text{Ext}_T(A, C)$. Notice that for each $\vec{\alpha} \in \vec{J}$ and $i < d$, $q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$, and $q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d)$.

We now construct a condition $q \in \mathbb{P}$ such that for each $\vec{\alpha} \in \vec{J}$, $q \leq p_{\vec{\alpha}}$. Let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair (i, γ) with $i < d$ and $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $j' < k^*$ such that $\vec{\delta}_{\vec{\alpha}}(j') = \gamma$. For any other $\vec{\beta} \in \vec{J}$ for which $\gamma \in \vec{\delta}_{\vec{\beta}}$, since the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is pairwise compatible by Lemma 27, it follows that $p_{\vec{\beta}}(i, \gamma)$ must equal $p_{\vec{\alpha}}(i, \gamma)$, which is exactly $t_{i,j'}^*$. Further note that in this case, $\vec{\delta}_{\vec{\beta}}(j')$ must also equal γ : If $j'' < k^*$ is the integer satisfying $\gamma = \vec{\delta}_{\vec{\beta}}(j'')$, then $t_{i,j''}^* = p_{\vec{\beta}}(i, \vec{\delta}_{\vec{\beta}}(j'')) = p_{\vec{\beta}}(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t_{i,j'}^*$, and hence $j'' = j'$. Let $q(i, \gamma)$ be the leftmost extension of $t_{i,j}^*$ in T . By the above argument, $q(i, \gamma)$ is well-defined.

Thus, $q(i, \gamma)$ is defined for each pair $(i, \gamma) \in d \times \vec{\delta}_q$. Define

$$(18) \quad q = \{q(d)\} \cup \{(i, \delta), q(i, \delta) : i < d, \delta \in \vec{\delta}_q\}.$$

Claim 5. For each $\vec{\alpha} \in \vec{J}$, $q \leq p_{\vec{\alpha}}$.

Proof. The proof begins with two general subclaims.

Subclaim (i). For each $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$ and $i < d$, if $\beta_i \neq \alpha_i$ then $\beta_i \notin \vec{\delta}_{\vec{\alpha}}$.

Proof. Suppose toward a contradiction that $\beta_i \in \vec{\delta}_{\vec{\alpha}}$. Then there is a $j < k^*$ such that $\beta_i = \delta_{\vec{\alpha}}(j)$. Since $\beta_i \neq \alpha_i = \delta_{\vec{\alpha}}(j_i)$, it must be that $j \neq j_i$. However, $\beta_i = \delta_{\vec{\beta}}(j_i) = \delta_{\vec{\alpha}}(j)$, so Claim 4 implies that $j_i = j$, a contradiction. \square

Subclaim (ii). If $\gamma, \delta \in \vec{\delta}_q$, $\gamma \in J_i$ and $\delta \in J_j$ for $i < j < d$, and the pair $\{q(i, \gamma), q(j, \delta)\}$ does not satisfy the Parallel 1's Criterion over A , then $\{\gamma, \delta\} \not\subseteq \vec{\delta}_{\vec{\alpha}}$ for any $\vec{\alpha} \in \vec{J}$.

Proof. If the pair $\{q(i, \gamma), q(j, \delta)\}$ does not satisfy the Parallel 1's Criterion over A , then the pair $\{\gamma, \delta\}$ does not appear as the i -th and j -th members of in any sequence in \vec{J} . Hence, for each $\vec{\alpha} \in \vec{J}$, either $\gamma \neq \alpha_i$ or $\delta \neq \alpha_j$. If $\gamma \neq \alpha_i$, then by Subclaim (i), $\gamma \notin \vec{\delta}_{\vec{\alpha}}$; likewise, if $\delta \neq \alpha_j$, then $\delta \notin \vec{\delta}_{\vec{\alpha}}$. Both cases show that $\{\gamma, \delta\} \not\subseteq \vec{\delta}_{\vec{\alpha}}$. \square

To prove the Claim, let $\vec{\alpha} \in \vec{J}$ be given. By our construction $q(d) \supseteq p_{\vec{\alpha}}(d)$, and for each pair $i < d$ and $\gamma \in \vec{\delta}_{\vec{\alpha}}$, $q(i, \gamma) \supseteq p_{\vec{\alpha}}(i, \gamma)$; so it only remains to show that (2)(ii) in the definition of the partial ordering on \mathbb{P} holds. Toward this end, suppose $\vec{\zeta}$ is a strictly increasing sequence in $[\vec{\delta}_{\vec{\alpha}}]^d$ and $\{p_{\vec{\alpha}}(i, \zeta_i) : i < d\} \cup \{p(d)\}$ is a member of $\text{Ext}_T(A, C)$. We need to show that $\{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$ is a member of $\text{Ext}_T(A, C)$.

If $\zeta_i \notin J_i$, then $q(i, \zeta_i)$ is the leftmost extension of $t_{i, j'}^*$ in $T \upharpoonright n_j$, where j' is the member of k^* such that $\zeta_i = \vec{\delta}_{\vec{\alpha}}(j')$, where $\vec{\alpha}$ is any member of \vec{N} for which $\zeta_i \in \vec{\delta}_{\vec{\alpha}}$. Thus, all such $q(i, \zeta_i)$ do not introduce any parallel 1's which were not already witnessed in A^+ . If $\zeta_i \in J_i$, then $q(i, \zeta_i)$ and $q(d)$ have parallel 1's only if s_i and s_d have parallel 1's. If $\zeta_i \in J_i$ and $\zeta_j \in J_j$, it follows from Subclaim (ii) that the pair $\{q(i, \zeta_i), q(j, \zeta_j)\}$ satisfies the Parallel 1's Criterion over A . Thus, $\{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$ satisfies the Parallel 1's Criterion over A and hence is a member of $\text{Ext}_T(A, C)$. Therefore, $q \leq p_{\vec{\alpha}}$. \square

To construct U_{m_j} , take an $r \leq q$ in \mathbb{P} which decides some $l_j \geq n_j$ in \dot{L}_d for which $c(\dot{b}_{\vec{\alpha}} \upharpoonright l_j) = \varepsilon^*$, for all $\vec{\alpha} \in \vec{J}$. Without loss of generality, we may assume that the nodes in the image of r have length l_j . Notice that since r forces $\dot{b}_{\vec{\alpha}} \upharpoonright l_j = \{r(i, \alpha_i) : i < d\} \cup \{r(d)\}$ for each $\vec{\alpha} \in \vec{J}$, and since the coloring c is defined in the ground model, it is simply true in the ground model that $c(\{r(i, \alpha_i) : i < d\} \cup \{r(d)\}) = \varepsilon^*$ for each $\vec{\alpha} \in \vec{J}$.

Extend the splitting node $q(d)$ in V_j to $r(d)$. For each $i < d$ and $\alpha_i \in J_i$, extend $q(i, \alpha_i)$ to $r(i, \alpha_i)$. Let V_j^- denote $V_j \setminus (\{q(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{q(d)\})$. For each node v in V_j^- , let v^* be the leftmost extension of v in $T \upharpoonright l_j$. Let

$$(19) \quad U_{m_j} = \{r(d)\} \cup \{r(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{v^* : v \in V_j^-\}.$$

This constructs U_{m_j} .

Claim 6. $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$ and every $D \in \text{Ext}_{U_{m_j}}(A, C)$ with $\max(D) \subseteq \max(U_{m_j})$ satisfies $c(D) = \varepsilon^*$.

Proof. The members of $\max(U_{m_j})$ are extensions of V_j , and for each $v \in V_j$ there is a unique $u_v \in \max(U_{m_j})$ end-extending v . First note that $r(d)$ is a splitting node extending the splitting node $q(d)$ of V_j ; so $u_{q(d)} = r(d)$. Y_j already is a member of $r_{m_j}[U_{m_j-1}, T]$, so Y_j is isomorphic to $r_{m_j-1}(T)$. For any pair $v, w \in V_j$ which has parallel 1's, this is already witnessed in U_{m_j-1} , since the critical node of V_j is a splitting node; such a pair may have any extensions u_v, u_w without violating the Parallel 1's Criterion for U_{m_j} . Thus, to prove that U_{m_j} is a member of $r_{m_j}[U_{m_j-1}, T]$, it suffices to show that for each pair $v, w \in V_j$ with no parallel 1's, their extensions $u_v, u_w \in \max(U_{m_j})$ also have no parallel 1's.

If $v \in V_j \setminus \bigcup_{i < d} X_i$, then u_v is the leftmost extension of v in $T \upharpoonright l_j$. For such v , if u_v has parallel 1's with some $u_w \in U_{m_j}$, then this is already witnessed in Y_j , and hence in U_{m_j-1} .

Now suppose v and w are nodes in $\bigcup_{i < d} X_i$ which do not have any parallel 1's. If v and w both extend s_i for some $i < d$, then that i must be i_0 and hence $v = w$; v and w cannot both extend the same s_i for any $i \in d \setminus \{i_0\}$, since s_i already has a 1 which would imply v and w have a parallel 1.

If $v \in X_i$ and $w \in X_j$ where $i \neq j$, then there is some $\vec{\alpha} \in \vec{J}$ such that $v = q(i, \alpha_i)$ and $w = q(j, \alpha_j)$. $\vec{\alpha} \in \vec{J}$ implies that $\{q(k, \alpha_k) : k < d\} \cup \{q(d)\}$ is in $\text{Ext}_T(A, C)$. Since $r \leq q$, $\{r(k, \alpha_k) : k < d\} \cup \{r(d)\}$ is in $\text{Ext}_T(A, C)$. Hence, since v and w have no parallel 1's, they certainly have no parallel 1's over A , so $r(i, \alpha_i)$ and $r(j, \alpha_j)$ also have no parallel 1's over A and therefore no parallel 1's at all, since $A \cup \{r(k, \alpha_k) : k < d\} \cup \{r(d)\}$ satisfies the Parallel 1's Criterion. Since $u_w = r(i, \alpha_i)$ and $u_v = r(j, \alpha_j)$, they have no parallel 1's.

Thus, every two members of V_j which have no parallel 1's have extensions in U_{m_j} which also have no parallel 1's. Therefore, U_{m_j} is a member of $r_{m_j}[U_{m_j-1}, T]$.

For each $X \in \text{Ext}_{U_{m_j}}(A, C)$ with $X \subseteq \max(U_{m_j})$ the truncation $A \cup \{x \upharpoonright |q(d)| : x \in X\}$ is a member of $\text{Ext}_{Y_j}(A, C)$. There corresponds a sequence $\vec{\alpha} \in \vec{J}$ such that $\{x \upharpoonright |q(d)| : x \in X\} = \{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$. Then $X = \{r(i, \alpha_i) : i < d\} \cup \{r(d)\}$, which has c -color ε^* . \square

Let $S = \bigcup_{j < \omega} U_{m_j}$. For each $X \in \text{Ext}_S(A, C)$, there corresponds a $j < \omega$ such that $X \in \text{Ext}_{U_{m_j}}(A, C)$ and $X \subseteq \max(U_{m_j})$. By Claim 9, $c(X) = \varepsilon^*$. Thus, S extends B^+ into T and satisfies the theorem. This concludes the proof of the theorem for Case (a).

Case (b). Let $X \in \text{Ext}_T(A, C)$ be fixed and let m_0 be least such that $X \subseteq r_{m_0}(T)$. Label the members of X as x_i , $i \leq d$, so that each $x_i \supseteq s_i$. Fix $r_{m_0-1}(T)$. Let n_0 be the index such that $c_{n_0}^T$ is the maximal coding node in $r_{m_0}(T)$. Let n_1 be the index of the least coding node in T strictly longer than $c_{n_0}^T$. For Case (b), back in Section 2 of the proof, choose the $p_{\vec{\alpha}}$ so that each $p_{\vec{\alpha}}(i, \alpha_i)$ extends x_i (in addition to extending c_i) and such that the length of $p_{\vec{\alpha}}(i, \alpha_i)$ is greater than $|c_{n_1}^T|$. This may be accomplished by choosing each $p_{\vec{\alpha}}^0(i, \alpha_i)$ to be x_i instead of c_i , and then choosing $p_{\vec{\alpha}}^1$ so that their lengths are longer than $|c_{n_1}^T|$. Then $t_i^* \supseteq x_i$ for each $i \leq d$, and $l^* > |c_{n_1}^T|$. We will build an $S \in [r_{m_0-1}(T), T]$ such that every member of $\text{Ext}_T(A, C)$ extending X has the same c -color.

Let $U_{m_0-1} = r_{m_0-1}(T)$. We will build U_{m_0} so that its maximal members extend the maximal members of $r_{m_0}(T)$, and hence each member of X is extended uniquely in U_{m_0} . Let $V_0 = \max(r_{m_0}(T))$. Then $X \subseteq V_0$. Let V_0^l and V_0^r denote those members v of V_0 such that the immediate extension of v is 0 or 1, respectively. Thus, $v \in V_0^l$ if and only if each extension of v of length at least $|c_{n_0}^T| + 1$ has passing number 0 at $c_{n_0}^T$; and $v \in V_0^r$ if and only if each extension of v of length at least $|c_{n_0}^T| + 1$ has passing number 1 at $c_{n_0}^T$.

Since $l^* > |c_{n_1}^T|$, the Passing Number Choice Lemma 23 guarantees that for each $v \in V_0^l \setminus X$, there is a member v^* extending v at level of t_d^* such that v^* has immediate successor 0 in T ; this v^* is the leftmost extension of v of length l^* , so any parallel 1's between v and an extension of some other member of V_0 must already be witnessed in U_{m_0-1} . Also, by the Lemma 23, for each $v \in V_0^r \setminus X$, there is a member v^* end-extending v at level of t_d^* such that v^* has immediate successor 1 in T ; this v^* is the leftmost extension of v until the maximal splitting node along this path before length l^* , at which point v^* takes the rightmost path. Thus, for any $v, w \in V_0$, if v^* and w^* have parallel 1's below the maximal splitting node in T below t_d^* , then v and w have parallel 1's.

Let

$$(20) \quad V^* = \{t_i^* : i \leq d\} \cup \{v^* : v \in r_{m_0}(T) \setminus X\}.$$

Claim 7. $U_{m_0-1} \cup V^*$ satisfies the Parallel 1's Criterion and $U_{m_0-1} \cup V^*$ is a member of $r_{m_0}[U_{m_0-1}, T]$.

Proof. Let y, z be members of $\max(V^*)$ which have parallel 1's. If these parallel 1's first occur in U_{m_0-1} , then there is nothing to check, since U_{m_0-1} satisfies the Parallel 1's Criterion; so suppose the first parallel 1's of y and z occur between the levels of $\max(U_{m_0-1})$ and V^* . It suffices to show that the immediate extensions of y and z are both 1, for then t_d^* is a coding node witnessing their parallel 1's.

First, note that y and z are both not members of $\{v^* : v \in V_0^l\}$. This follows since each $v^* \in V^* \setminus \{t_i^* : i \leq d\}$ which extends a member v of V_0^l is the leftmost extension of v to the level of l^* and has passing number 0 at t_d^* ; such a v^* has no parallel 1's between the levels of $|c_{n_0}^T|$ and $l^* + 1$ with any other node in V^* with which it has no parallel 1 below $|c_{n_0}^T|$.

If both y, z are in $\{v^* : v \in V_0^r \setminus X\}$, then both y and z have passing number 1 at t_d^* . Suppose $y, z \in \{t_i^* : i \leq d\}$. Then since $\{t_i^* : i \leq d\} \in \text{Ext}_T(A, C)$, the parallel 1's of y and z must be witnessed by $A \cup \{t_i^* : i \leq d\}$. Since we are assuming y and z have no parallel 1's below $|c_{n_0}^T|$, they must both have passing number 1 at t_d^* .

Lastly, suppose $y \in \{v^* : v \in V_0^r \setminus X\}$ and $z = t_i^*$ for some $i \leq d$. Since y is chosen by taking the rightmost extension of $\text{splitpred}_T(v_0^*)$ to the level of t_d^* , where v_0^* denotes the leftmost extension of v to level $|t_d^*|$, y has no parallel 1's with z below the level of $\text{splitpred}_T(v_0^*)$. Thus, any parallel 1's y and z have must occur above the longest splitting node below $|t_d^*|$. Since we are assuming that y and z have a parallel 1 and that the least parallel 1 occurs above $\max(U_{m_0-1})$, it must be the case that z also has passing number 1 at t_d^* . (For if not, then that means that z chose the leftmost extension above $\text{splitpred}_T(z)$, in which case z cannot have a parallel 1 with y at all.)

Thus, $U_{m_0-1} \cup V^*$ satisfies the Parallel 1's Criterion. Furthermore, is it simple to check that for each $y \in \max(V^*)$, y has passing number 1 at t_d^* if and only if y has passing number 1 at $c_{n_0}^T$. Thus $U_{m_0-1} \cup V^* \stackrel{s}{\sim} r_{m_0}(T)$ and hence $U_{m_0-1} \cup V^* \in r_{m_0}[U_{m_0-1}, T]$. \square

Define $U_{m_0} = U_{m_0-1} \cup V^*$. Let $Y \in [U_{m_0}, T]$ and define $U_{m_1-1} = r_{m_1-1}(Y)$. Notice that $\{t_i^* : i \leq d\}$ is the only member of $\text{Ext}_{U_{m_1-1}}(A, C)$, and it has c -color ε^* .

Induction Assumption. Assume $1 \leq j < \omega$ and we have constructed U_{m_j-1} so that every member of $\text{Ext}_{U_{m_j-1}}(A, C)$ is colored ε^* by c .

Fix some $Y_j \in r_{m_j}[U_{m_j-1}, T]$. Let V_j denote $\max(Y_j)$, and let n_j denote the length of the members of V_j . Note that n_j is a member of L . The nodes in V_j will not be in the tree S we are constructing; rather, we will extend the nodes in V_j to construct $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$. Let $q(d)$ be the coding node in V_j . Let I_0 denote the set of $i < d$ for which t_i^* has passing number 0 at t_d^* ; let I_1 denote the set of $i < d$ for which t_i^* has passing number 1 at t_d^* . For each $i \in I_0$, let X_i be the set of nodes u in $V_j \cap T_i$ such that u has passing number 0 at $q(d)$; for each $i \in I_1$, let X_i be the set of nodes u in $V_j \cap T_i$ such that u has passing number 1 at $q(d)$.

We now construct a condition q similarly to, but not exactly the same as, Case (a). For each $i < d$, let K_i'' be a subset of K_i' with the same size as X_i . For each $i < d$, label the nodes in X_i as $\{x(i, \delta) : \delta \in K_i''\}$. Let \vec{J} denote the set of those $\langle \alpha_0, \dots, \alpha_{d-1} \rangle \in \prod_{i < d} K_i''$

such that the set $\{x(i, \alpha_i) : i < d\} \cup \{q(d)\}$ is in $\text{Ext}_T(A, C)$. Notice that for each $\vec{\alpha} \in \vec{J}$ and $i < d$, $x(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$, and $q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d)$. Let $J_i = \{\delta : \exists \vec{\alpha} \in \vec{J} (\delta = \alpha_i)\}$.

Let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair (i, γ) with $\gamma \in J_i$, define $q(i, \gamma) = x(i, \gamma)$. For each pair (i, γ) with $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $j' < k^*$ such that $\vec{\delta}_{\vec{\alpha}}(j') = \gamma$. It follows by Lemma 27 for any $\vec{\beta} \in \vec{J}$ for which $\gamma \in \vec{\delta}_{\vec{\beta}}$, $p_{\vec{\beta}}(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t_{i,j'}^*$. By the same argument as for Case (a), $\vec{\delta}_{\vec{\beta}}(j')$ must also equal γ . For $i \in I_0$, let $q(i, \gamma)$ be the leftmost extension of $t_{i,j'}^*$ in T . This will have passing number 0 at $q(d)$, and any parallel 1's between this node and another node in V_j must be witnessed at or below t_d^* . For $i \in I_1$, let $q(i, \gamma)$ be the extension of $t_{i,j}^*$ leftmost until the level of the maximal coding node in T below $q(d)$; then extend to the next splitting node and take the right branch to the level of $q(d)$. Let this node be $q(i, \gamma)$. This will have passing number 1 at $q(d)$. Any parallel 1's between this node and another must be either witnessed by $q(d)$ or else at or below t_d^* . Define

$$(21) \quad q = \{q(d)\} \cup \{(i, \delta), q(i, \delta) : i < d, \delta \in \vec{\delta}_q\}.$$

Claim 8. For each $\vec{\alpha} \in \vec{J}$, $q \leq p_{\vec{\alpha}}$.

Proof. Subclaim (i) and its proof are the same as in Case (a).

Subclaim (i). For each $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$ and $i < d$, if $\beta_i \neq \alpha_i$ then $\beta_i \notin \vec{\delta}_{\vec{\alpha}}$.

Subclaim (ii). If $\gamma, \delta \in \vec{\delta}_q$, $\gamma \in J_i$ and $\delta \in J_j$ for $i < j < d$, and the pair $\{q(i, \gamma), q(j, \delta)\}$ has no parallel 1's below A_e , has parallel 1's between the level of A_e and $q(d)$, and at least one of them has passing number 0 at $q(d)$, then $\{\gamma, \delta\} \not\subseteq \vec{\delta}_{\vec{\alpha}}$ for any $\vec{\alpha} \in \vec{J}$.

Proof. If $\{q(i, \gamma), q(j, \delta)\}$ are as in the hypotheses, then the pair does not appear as the i -th and j -th members in any sequence in \vec{J} . As in Case (a), Subclaim (i) implies that $\{\gamma, \delta\} \not\subseteq \vec{\delta}_{\vec{\alpha}}$ for any $\vec{\alpha} \in \vec{J}$. \square

To prove the Claim, let $\vec{\alpha} \in \vec{J}$ be given. By our construction $q(d) \supseteq p_{\vec{\alpha}}(d)$, and for each pair $(i, \gamma) \in d \times \vec{\delta}_{\vec{\alpha}}$, $q(i, \gamma) \supseteq p_{\vec{\alpha}}(i, \gamma)$; so it only remains to show that (2)(ii) in the definition of the partial ordering on \mathbb{P} holds. Toward this end, suppose $\vec{\zeta}$ is a strictly increasing sequence in $[\vec{\delta}_{\vec{\alpha}}]^d$ and $\{p_{\vec{\alpha}}(i, \zeta_i) : i < d\} \cup \{p(d)\}$ is a member of $\text{Ext}_T(A, C)$. We claim that $\{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$ is a member of $\text{Ext}_T(A, C)$. It suffices to show that $A \cup \{q_{\vec{\alpha}}(i, \zeta_i) : i < d\} \cup \{q(d)\}$ satisfies the Parallel 1's Criterion, since correct passing numbers at the level of $q(d)$ were taken care of when the $q(i, \gamma)$ were chosen.

For each $i < j < d$ such that $q(i, \zeta_i)$ and $q(j, \zeta_j)$ have no parallel 1's in A^+ and no parallel 1's at $q(d)$, $q(i, \zeta_i)$ and $q(j, \zeta_j)$ must not have any parallel 1's, by Subclaim (ii).

Now we want to show that if $q(i, \zeta_i)$ and $q(d)$ have parallel 1's, then this occurred in A^+ and so was witnessed in A . If $\zeta_i \in J_i$, then ζ_i is the i -th member of some sequence $\vec{\alpha} \in \vec{J}$. Hence by definition of \vec{J} , $q(i, \zeta_i)$ and $q(d)$ satisfy the Parallel 1's Criterion in $A \cup \{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$.

Now suppose that $\zeta_i \in \vec{\delta}_{\vec{\alpha}} \setminus J_i$. Then $q(i, \zeta_i)$ extends $t_{i,j'}$ for some j' . If $i \in I_0$, then $q(i, \zeta_i)$ was the leftmost extension of $t_{i,j'}$, so if $q(i, \zeta_i)$ and $q(d)$ have parallel 1's then they must have parallel 1's strictly below the level t_d^* ; that is, $t_{i,j'}$ and t_d^* have a parallel 1. (Since $q(d)(|t_d^*|) = 0$, $q(i, \zeta_i)$ and $q(d)$ cannot have parallel 1's at level $|t_d^*|$.) Since $\vec{\zeta} \in \vec{J}$, $A \cup \{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$ satisfies the Parallel 1's Criterion, the parallel 1 between

$q(i, \zeta_i)$ and $q(d)$ must be witnessed in A . If $i \in I_1$, then since $q(d)$ is a coding node and the immediate extension of $q(i, \zeta_i)$ is 1, $q(d)$ has no parallel 1's with $q(i, \zeta_i)$.

Thus, $A \cup \{q(i, \zeta_i) : i < d\} \cup \{q(d)\}$ satisfies the Parallel 1's Criterion. Therefore, $q \leq p_{\vec{\alpha}}$. \square

To construct U_{m_j} , we extend each node in V_j uniquely in such a manner so that these extensions along with $U_{m_{j-1}}$ form a member of $r_{m_j}[B, T]$. Take an $r \leq q$ in \mathbb{P} which decides some $l_j \geq n_j$ in \dot{L}_d for which $c(\dot{b}_{\vec{\alpha}} \upharpoonright l_j) = \varepsilon^*$, for all $\vec{\alpha} \in \vec{J}$. Without loss of generality, we may assume that the nodes in the image of r have length l_j . Extend the coding node $q(d)$ in V_j to $r(d)$. For each $i < d$ and $\alpha_i \in J_i$, extend $q(i, \alpha_i)$ to $r(i, \alpha_i)$. Let V_j^- denote $V_j \setminus (\{q(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{q(d)\})$. For each node v in V_j^- such that the passing number of v at $q(d)$ is 0, let v^* be the leftmost extension of v in $T \upharpoonright l_j$. For each node v in V_j^- such that the passing number of v at $q(d)$ is 1, extend v leftmost to v' of length $r(d)$, then let v^* be the right extension of $\text{splitpred}_T(v')$ to the level of $r(d)$. Then v^* will have passing number 1 at $r(d)$. This preserves passing number: The immediate extension of v^* is the same integer as the immediate extension of v .

Let

$$(22) \quad U_{m_j} = \{r(d)\} \cup \{r(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{v^* : v \in V_j^-\}.$$

Claim 9. U_{m_j} is a member of $r_{m_j}[U_{m_{j-1}}, T]$, and $c(D) = \varepsilon^*$ for each $D \in \text{Ext}_{U_{m_j}}(A, C)$.

Remark. There is a difference between showing that $q \leq p_{\vec{\alpha}}$ for all $\vec{\alpha} \in \vec{J}$, in the previous claim, and showing that $U_{m_j} \in r_{m_j}[U_{m_{j-1}}, T]$, below. To show $q \leq p_{\vec{\alpha}}$, we had to consider $t_{i,j'}$'s which will not actually be in \hat{U}_{m_j} to ensure satisfaction of the partial ordering on \mathbb{P} . Forming U_{m_j} involves extending only the members of V_j , while ensuring that it satisfies the Parallel 1's Criterion and that the passing numbers are correct in order to be a member of $r_{m_j}[U_{m_{j-1}}, T]$.

Proof. To show that U_{m_j} is a member of $r_{m_j}[U_{m_{j-1}}, T]$, it suffices to show that U_{m_j} satisfies the Parallel 1's Criterion and that for each $v \in V_j$, its unique extension to a member of $\max(U_{m_j})$ has immediate extension equal to the immediate extension of v . This ensures that the passing numbers are correct to be a member of $r_{m_j}[U_{m_{j-1}}, T]$. For $v \in V_j$, letting v^* denote its extension in $\max(U_{m_j})$, we have already shown that the passing number of v at $q(d)$ is the same integer as the passing number of v^* at $r(d)$. So to show that U_{m_j} satisfies the Parallel 1's Criterion, since $r_{m_{j-1}}(U_{m_j}) = r_{m_{j-1}}(Y_j)$, it suffices to show that each pair of nodes y, z in $\max(U_{m_j})$ with first parallel 1's above $\max(U_{m_{j-1}})$, is witnessed by $r(d)$; that is, they both have passing number 1 at $r(d)$.

Let n_* denote $|u| + 1$ for $u \in \max(U_{m_{j-1}})$, and let n^* denote the length of members of V_j .

Subclaim 1. Let $v \in V_j^-$ such that v^* has passing number 0 at $r(d)$. Then for any $x \in \max(U_{m_j})$, if v and $x \upharpoonright n_*$ have no parallel 1's then v^* and x have no parallel 1's.

Proof. Suppose $x \in \max(U_{m_j})$ and $x \upharpoonright n_*$ has no parallel 1's with $v \upharpoonright n_*$. Since v^* has passing number 0 at $r(d)$, it is the leftmost extension of $v := v^* \upharpoonright n^*$. Since v^* is the leftmost extension of v , it must be the case that v has passing number 0 at level n^* . Since $U_{m_{j-1}} \cup V_j \in r_{m_j}[U_{m_{j-1}}, T]$, we know that $U_{m_{j-1}} \cup V_j$ satisfies the Parallel 1's Criterion. Therefore, any parallel 1's between v and $x \upharpoonright n^*$ must be witnessed in $U_{m_{j-1}}$. Since members

of U_{m_j-1} all have length less than or equal to n_* , and $x \upharpoonright n_*$ has no parallel 1's with $v \upharpoonright n_*$, it follows from Fact 24 that v^* and x have no parallel 1's. \square

Thus, for the rest of the proof of the present claim, we make the following

Assumption. Let y, z be nodes in $\max(U_{m_j})$ with parallel 1's and such that their least parallel 1's occur above n_* .

We will show that they both must have passing number 1 at $r(d)$. The rest of the proof of this claim involves investigating the various cases that arise.

Subclaim 2. If y, z are both in V_j^- , then y and z must both have passing number 1 at $r(d)$.

Proof. Suppose $y = v^*$ for some member v of V_j^- . If v has passing number 0 at $q(d)$, then by Subclaim 1, in order for y to have a parallel 1 with another member x of U_{m_j} , v and $x \upharpoonright n_*$ must have a parallel 1. Since we are assuming the first parallel 1 between y and z occurs above n_* , it follows that v must have passing number 1 at $q(d)$. Therefore, y has passing number 1 at $r(d)$, by construction of v^* . As the same argument works for z , it follows that if both y, z are in V_j^- , then they both have passing numbers 1 at $r(d)$. \square

Subclaim 3. Suppose both y and z are in $\{r(i, \alpha_i) : i < d, \alpha_i \in J_i\}$. Then y and z have passing number 1 at $r(d)$.

Proof. Suppose $y = r(i, \gamma)$ and $z = r(j, \delta)$ for some $i, j < d$, $\gamma \in J_i$, and $\delta \in J_j$. If $i = j$, then both y and z extend s_i , which is in U_{m_0-1} . If $i = i_0$, then $y = z$ and are members of $0^{<\omega}$, hence they do not have parallel 1's, a contradiction to our assumption on y and z . If $i \neq i_0$, then s_i has some 1 in it, so y and z have a parallel 1 below n_* , again a contradiction.

Now assume $i \neq j$. Since y and z have no parallel 1's at or below n_* , if y and z do not have parallel 1's at n^* , then their passing numbers at $q(d)$ are both 0. Then both i, j are in I_0 . In this case, since y and z have no parallel 1's below n^* , it must be that s_i and s_j have no parallel 1's. Then all extensions of s_i and s_j into a member of $\text{Ext}_T(A, C)$ add no parallel 1's and have passing numbers 0 at the coding node in the copy of C . Therefore, the passing numbers of y and z at $r(d)$ are both 0. But this implies y and z have no parallel 1's, contradicting our assumption on y and z . Therefore, y and z must have parallel 1's at n^* and hence, y and z also have passing number 1 at $r(d)$. \square

Subclaim 4. Suppose $y \in \{r(i, \alpha_i) : i < d, \alpha_i \in J_i\}$. Then y and $r(d)$ have no parallel 1's. Hence, z cannot equal $r(d)$.

Proof. If y has passing number 1 at $r(d)$, then y and $r(d)$ have no parallel 1's since $r(d)$ is a coding node, contradicting our assumption. So y must have passing number 0 at $r(d)$. It follows that $y \upharpoonright n^*$ has passing number 0 at $q(d)$. Since $Y_j \in r_{m_j}[U_{m_j-1}, T]$ and y and z are assumed to have no parallel 1's at or below n_* , it follows that $y \upharpoonright n^*$ and $q(d)$ must have no parallel 1's. Then by the definition of the forcing \mathbb{P} , y and $r(d)$ must also have no parallel 1's. (That is, there is a member of $\text{Ext}_T(A, C)$ containing both $y \upharpoonright n^*$ and $q(d)$. The configuration of these two must be preserved in their extension to make another member of $\text{Ext}_T(A, C)$, so if $y \upharpoonright n^*$ and $q(d)$ have no parallel 1's, then neither do y and $r(d)$.) Since $r(d)$ has immediate extension 0, y and $r(d)$ have no parallel 1's at or below $r(d)$. \square

Subclaim 5. If y extends a member of V_j^- , then y and $r(d)$ have no parallel 1's. Hence, z cannot equal $r(d)$.

Proof. Since we are assuming that y and z have no parallel 1's at or below n_* , and since $Y_j \in r_{m_j}[U_{m_j-1}, T]$ and $q(d)$ is a coding node, $q(d)$ has immediate extension 0, so $y \upharpoonright (n^* + 1)$ and $r(d) \upharpoonright (n^* + 1)$ have no parallel 1's. If y is the leftmost extension of $y \upharpoonright n^*$, then by Subclaim 1, y has no parallel 1's with $r(d)$. If y has passing number 1 at $r(d)$, then $\text{splitpred}_T(y)$ is the leftmost extension of $y \upharpoonright n^*$. Since $r(d) \upharpoonright n^*$ and $y \upharpoonright n^*$ have no parallel 1's, Subclaim 1 implies that $\text{splitpred}_T(y)$ has no parallel 1's with $r(d) \upharpoonright |\text{splitpred}_T(y)|$. Since the immediate extension of $r(d)$ is 0 and $r(d)$ is a coding node, it follows that $r(d)$ and y have no parallel 1's. \square

Subclaim 6. Suppose $y \in \{r(i, \alpha_i) : i < d, \alpha_i \in J_i\}$ and z extends a member of V_j^- . Then y and z both have passing number 1 at $r(d)$.

Proof. Let $i < d, \gamma \in J_i$ be such that y extends $q(i, \gamma)$. By Subclaim 1, z must have passing number 1 at $r(d)$ since y and z have parallel 1's, so it remains to show that also y has passing number 1 at $r(d)$. It suffices to show that $i \in I_1$.

Suppose toward a contradiction that $i \in I_0$. Then y has passing number 0 at $r(d)$. Therefore, $y \upharpoonright n^*$ also has passing number 0 at $q(d)$. Since y and z have no parallel 1's below n_* and y has passing number 0 at n^* , y and z must also have no parallel 1's below $n^* + 1$, since $U_{m_j-1} \cup V_j$ satisfies the Parallel 1's Criterion, and $q(d)$ is the only coding node that could witness a new parallel 1 in the interval $[n_*, n^*]$. Between n^* and $|\text{splitpred}_T(z)|$, z follows the leftmost path extending $z \upharpoonright n^*$, so in this interval, z does not have any parallel 1's with y , since they have no parallel 1's below $n^* + 1$. Since $i \in I_0$, y takes the leftmost path in the interval between the longest coding node in T below $r(d)$ and $r(d)$; so in particular, y has no parallel 1's with z in the interval between $|\text{splitpred}_T(z)|$ and l_j . Since y has passing number 0 at $r(d)$, y and z must have no parallel 1's at all, a contradiction. Therefore, i must be in I_1 and hence y has passing number 1 at $r(d)$ as claimed. \square

This concludes the proof of Claim 9. \square

By Claim 9, $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$. As in Case (a), for each $Z \in \text{Ext}_{U_{m_j}}(A, C; X)$ with $Z \subseteq \max(U_{m_j})$, the truncation $A \cup \{x \upharpoonright |q(d)| : x \in Z\}$ is a member of $\text{Ext}_{Y_j}(A, C)$. There corresponds a sequence $\vec{\alpha} \in \vec{J}$ such that $\{z \upharpoonright |q(d)| : z \in Z\} = \{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$. Then $Z = \{r(i, \alpha_i) : i < d\} \cup \{r(d)\}$, which r forces to have c -color ε^* . Since $c(Z)$ is defined in the ground model, $c(Z) = \varepsilon^*$.

To finish the proof of the theorem for Case (b), Define $S = \bigcup_{j < \omega} U_{m_j}$. For each $Z \in \text{Ext}_S(A, C; X)$, there is a $j < \omega$ such that $Z \in \text{Ext}_{U_{m_j}}(A, C)$ and each member of $\max(U_{m_j})$ extending X has c -color ε^* . Thus, S extends B^+ into T and satisfies the theorem for Case (b).

This concludes the proof of the theorem. \square

5. RAMSEY THEOREM FOR FINITE TREES SATISFYING THE STRICT PARALLEL 1'S CRITERION

Our first Ramsey theorem for finite trees appears in this section as Theorem 29. It holds for a restricted class of copies of a given finite tree. Given a finite tree Z satisfying the Parallel 1's Criterion, and given a coloring of $\text{Sim}_T^s(Z)$ into finitely many colors, we obtain a strong coding tree $S \leq T$ with one color for all trees Y strongly similar to Z , with the

restriction that Y satisfies the Strict Parallel 1's Criterion inside S , defined below. This is a step toward the main result of this paper, Theorem 44, which will be proved in Section 7.

Let T be a strong coding tree. An *interval* of T is the set of levels between successive coding nodes in T ; thus $[0, |c_0^T|]$ is the least interval in T , and $(|c_n^T|, |c_{n+1}^T|]$ is the $n + 1$ -st interval in T . Let Z be a finite subtree of T . We say that l is a *minimal level of a new set of parallel 1's in Z* if $|z(l) : z \in \max(Z)| \geq 2$ and for all $l' < l$, there is at least one $z \in \max(Z)$ such that $z(l) = 1$ and $z(l') = 0$.

Definition 28 (Strict Parallel 1's Criterion). Z satisfies the *Strict Parallel 1's Criterion* in T if Z satisfies the Parallel 1's Criterion and moreover, whenever l is a minimal level of a new set of parallel 1's in Z , for the minimal coding node c_n^T such that $|c_n^T| \geq l$, l is in the interval $(|c_{n-1}^T|, |c_n^T|]$, and $\text{splitpred}_T(\{z \upharpoonright |c_n^T| : z \in Z\})$ satisfies the Parallel 1's Criterion.

Given a finite tree Z satisfying the Parallel 1's Criterion and contained in a strong coding tree T , let $\text{Sim}_T^{SP}(Z)$ denote the set Y of all finite subtrees of T such that Y is strongly similar to Z and moreover, satisfies the Strict Parallel 1's Criterion in T .

We point out that in Definition 28, it follows that for each $z \in Z$ of length greater than l , it is also the case that $|z| \geq |c_n^T|$, and moreover, $z(l) = 1$ if and only if $z(|c_n^T|) = 1$, since new sets of parallel 1's never occur before the length of the longest splitting node in an interval of T .

Theorem 29. *Let T be a strong coding tree, and let Z be a finite subtree of T satisfying the Parallel 1's Criterion. For any coloring of $\text{Sim}_T^s(Z)$ into finitely many colors, there is a strong coding tree $S \leq T$ such that all $Y \in \text{Sim}_S^{SP}(Z)$ have the same color.*

The theorem will be proved via four lemmas and then doing a fusion argument.

We begin with the definitions building up to the definition of end-homogeneity (Definition 31). Let A be a fixed finite subtree of a strong coding tree T such that A satisfies the Parallel 1's Criterion. Let A_e be a subset of A^+ . Suppose X is a level subset of a strong coding tree T , X contains a coding node, and each member of A_e is extended by a unique member of X . Let $d + 1 = |X|$ and label the nodes in X as x_i , $i \leq d$. Let I_1 denote the set of $i < d$ such that the immediate extension of x_i in T is 1, and let I_0 denote the set of $i \leq d$ such that the immediate extension of x_i in T is 0. Let $C = A \cup X$. Since X contains a coding node, the immediate successors of the members of X are well-defined; these are implicitly considered when determining whether or not a level set Y is in $\text{Ext}_T(A, C)$, and whether or not $A \cup Y$ satisfies the Parallel 1's Criterion.

Define $\text{Ext}_T^{SP}(A, C)$ to be the set of those $Y \in \text{Ext}_T(A, C)$ such that $A \cup Y$ satisfies the Strict Parallel 1's Criterion in T . Notice that if the parallel 1's in X are already witnessed in A , then $\text{Ext}_T^{SP}(A, C)$ is equal to $\text{Ext}_T(A, C)$. If there are parallel 1's in X not witnessed in A , then $Y \in \text{Ext}_T^{SP}(A, C)$ if and only if $Y \in \text{Ext}_T(A, C)$ and for the minimal l such that $\{i < d : y_i(l) = 1\} = I_1$, $A \cup \{\text{splitpred}_T(y_i \upharpoonright l) : i \in I_1\} \cup \{y_i \upharpoonright l : i \in I_0\}$ satisfies the Parallel 1's Criterion.

Next, we define the notion of minimal pre-extension of A to a copy of C . This notion will be used in the next lemma to obtain a strong form of end-homogeneity for the case when $\max(C)$ has a coding node.

Definition 30 (Minimal pre-extension of A to a copy of C). Let $\{u_i : i \leq d\}$ be any level set end-extending A_e such that, letting $l = |u_i|$ for $i \leq d$, l is the length of some coding node in T and

- (i) Letting u_i^+ denote the immediate extension of u_i in T , $\{i \leq d : u_i^+(l) = 1\} = I_1$; and
- (ii) $A \cup \{\text{splitpred}_T(u_i) : i \in I_1\} \cup \{u_i : i \in I_0\}$ satisfies the Parallel 1's Criterion.

We call $\{u_i : i \leq d\}$ a *minimal pre-extension (in T) of A to a copy of C* , or simply a *minimal pre-extension* when T , A and C are clear.

Notice that each minimal pre-extension in T of A to a copy of C can be extended to a member of $\text{Ext}_T^{SP}(A, C)$. Given a minimal pre-extension U , let $\text{Ext}_T^{SP}(A, C; U)$ denote the set of all $Y \in \text{Ext}_T^{SP}(A, C)$ such that Y end-extends U .

Remark. Notice that $\text{Ext}_T^{SP}(A, C; U)$ is exactly the set of all $Y \in \text{Ext}_T(A, C)$ which extend U . We leave the SP in the superscript to remind the reader that all members of $\text{Ext}_T(A, C)$ extending a minimal pre-extension U must actually satisfy the Strict Parallel 1's Criterion.

Definition 31 (End-homogeneity). A coloring on $\text{Ext}_T^{SP}(A, C)$ is *end-homogeneous* if for each minimal pre-extension $U = \{u_i : i \leq d\}$ of A to a copy of C in T , every member of $\text{Ext}_T^{SP}(A, C; U)$ has the same color.

Lemma 32. *Let T be a strong coding tree, A a finite subtree of T satisfying the Parallel 1's Criterion, X a level subset of T extending the nodes in A_e (which is a subset of A^+) with a coding node in X such that $C := A \cup X$ satisfies the Parallel 1's Criterion. Let k be minimal such that $\max(A) \subseteq r_k(T)$, and let c be a coloring of $\text{Ext}_T(A, C)$ into two colors. Then there is a $T' \in [r_k(T), T]$ such that c is end-homogeneous on $\text{Ext}_{T'}^{SP}(A, C)$.*

Proof. Let A_e denote those members of A^+ which are extended to some node in X . Let s_i , $i \leq d$, denote the nodes in A_e , where s_d is the node which is extended to the coding node in X . Then s_d is the node that the coding node in Y extends, for any $Y \in \text{Ext}_T(A, C)$. Let I_1 denote the set of all $i \leq d$ such that the node $c_i \in X$ extending s_i has passing number 1 in T at c_d . Let $I_0 = (d+1) \setminus I_1$. Recall that each member $X \in \text{Ext}_T(A, C)$ has the property that, letting x_i denote the member of X extending s_i , x_i has immediate extension 1 in T if and only if $i \in I_1$.

Let $(n_i)_{i < \omega}$, be those integers greater than k such that there is a minimal pre-extension of A to a copy of C from among the maximal nodes in $r_{n_i}(T)$. Note that each of these $r_{n_i}(T)$ contains a coding node. Let $r_{n_0-1}(T') = r_{n_0-1}(T)$. Enumerate the collection of all minimal pre-extensions of A to C from among the maximal nodes in $r_{n_0}(T)$, say U_0, \dots, U_q . We will build $r_{n_0}(T_0) \in r_{n_0}[r_{n_0-1}(T), T]$ so that each node in $\max(r_{n_0}(T_0))$ extends a unique node in $\max(r_{n_0}(T))$.

Extend the nodes in U_0 to some $V_0 \in \text{Ext}_T(A, C; U_0)$. Notice that every member of $\text{Ext}_T(A, C; U_0)$ is automatically a member of $\text{Ext}_S^{SP}(A, C)$ for any $S \in [r_{n_0-1}(T), T]$. Apply Case (b) from Theorem 22, replacing X in the statement of that theorem with V_0 . To do this, we fix $r_{n_0-1}(T)$, extend the nodes in U_0 to the nodes in V_0 , and extend the nodes in $\max(r_{n_0}(T)) \setminus U_0$ via their leftmost extensions to the length of the nodes in V_0 . Let W_0 denote this level set. Then W_0 is a level set end-extending the level set $\max(r_{n_0}(T))$. Notice that all parallel 1's in W_0 are already witnessed in $r_{n_0}(T)$. Apply Case (b) of Theorem 22 above the level set W_0 to obtain a strong coding tree $T_0 \in [r_{n_0-1}(T), T]$ such that the coloring on $\text{Ext}_{T_0}^{SP}(A, C)$ is end-homogeneous above U_0 ; each member of $\text{Ext}_{T_0}(A, C; V_0)$ has the same color. Since V_0 extends U_0 and there are no splitting nodes in V_0 above U_0 , T_0 is homogeneous on $\text{Ext}_{T_0}^{SP}(A, C; U_0)$.

Continuing in this manner, suppose $p < q$ and we have constructed $T_p \in [r_{n_0-1}(T), T]$ such that the coloring is homogeneous on $\text{Ext}_{T_p}^{SP}(A, C; U_p)$. Extend the nodes in U_{p+1} to some $V_{p+1} \in \text{Ext}_{T_p}(A, C; U_{p+1})$ which end-extends the nodes above U_{p+1} in W_p . Let W_{p+1} consist of V_{p+1} along with the set of leftmost extensions of the nodes in $W_p \setminus U_{p+1}$ to the level of the nodes in V_{p+1} . Apply Case (b) of Theorem 22 in T_p above the nodes in V_{p+1} to obtain $T_{p+1} \in [r_{n_0-1}(T), T_p]$ such that the coloring c is homogeneous on $\text{Ext}_{T_{p+1}}(A, C; V_{p+1})$, and hence is homogeneous on $\text{Ext}_{T_{p+1}}^{SP}(A, C; U_{p+1})$.

At the end of this process, let $T^0 = T_q$. By our construction, $T^0 \in [r_{n_0-1}(T), T]$ and satisfies that for each $p \leq q$, $\text{Ext}_{T^0}^{SP}(A, C; U_p)$ is homogeneous. Since each the nodes in $\max(r_{n_0}(T))$ have unique extensions to $\max(r_{n_0}(T^0))$, T^0 is end-homogeneous above each minimal pre-extension in $\max(r_{n_0}(T^0))$ of A to a copy of C .

By induction on $i < \omega$, given T^i which is end-homogeneous for all minimal pre-extensions in $\max(r_{n_i}(T^i))$ of A to a copy of C , we obtain $T^{i+1} \in [r_{n_{i+1}-1}(T^i), T^i]$ such that for minimal pre-extension V of A to a copy of C in T^i , we have one color for all members of $\text{Ext}_{T^{i+1}}^{SP}(A, C; V)$. Define

$$(23) \quad T' = \bigcup_{i < \omega} r_{n_{i+1}-1}(T^i).$$

Then T' satisfies the Lemma. □

Recall that for a strong coding tree T , $\langle c_n^T : n < \omega \rangle$ denotes the enumeration of the coding nodes in T in order of increasing length. Let c_{-1}^T denote the empty sequence.

Let A be a finite subtree of a strong coding tree T and let l be the maximum length of the nodes in A . Suppose there is a splitting node in A of length l , and let A^+ denote the set of immediate extensions in T of those $s \in A$ such that $|s| = l$. We say that A *has no pre-determined new parallel 1's in T* if the following hold: Letting n be minimal such that $|c_n^T| \geq l$, there is an extension A^* in T of A^+ to the next coding node $|c_n^T|$ level in T such that any parallel 1's in A^* at level $|c_n^T|$ are already witnessed in A .

Definition 33 (Thin subtree). We shall say that a finite subtree A of a strong coding tree T is *thin* in T if each critical node of A comes from within a different interval of A , and moreover, in each interval of T in which a splitting node of A occurs, all other nodes in A in that interval take the leftmost path above any splitting node through which they pass. Similarly, if B is an initial segment of a strong coding tree, we say that a subtree A of B is *thin* in B if each critical node of A comes from a different interval of B .

The next lemma provides a means for homogenizing the end-homogeneity for the Case (b), when $\max(C)$ contains a coding node, to obtain one color for all members $Y \in \text{Ext}_S^{SP}(A, C)$, provided that A and $A \cup Y$ are thin in S . The arguments are often similar to those of Case (a) of Theorem 22, but sufficiently different to warrant a proof. In a later lemma, we will diagonalize to obtain a strong coding tree S' such that, for each copy A' of A which is thin in S' , each $Y \in \text{Ext}_{S'}^{SP}(A', C)$ such that $A' \cup Y$ is thin in S' has the same c -color. Then we will take a strong coding tree S'' which is thin inside S' so that for each copy A'' of A in S'' , $\text{Ext}_{S''}^{SP}(A'', C)$ is homogeneous.

Lemma 34. *Let A, A_e, C, c be as in Lemma 32, assume $\max(C)$ contains a coding node, and suppose that c is end-homogeneous on $\text{Ext}_{T'}^{SP}(A, C)$. Suppose that $B = r_k(T')$ for some*

k such that $\max(B)$ contains a coding node. Suppose A is thin in B and the maximum level of A is contained in the maximum interval of B .

Then there is an $S \in [B, T']$ such that c is homogeneous on the set of all $Y \in \text{Ext}_S^{SP}(A, C)$ such that $A \cup Y$ is thin in S .

Proof. We begin by pointing out that A being thin in B implies that A has no pre-determined new parallel 1's in B , and hence also no pre-determined new parallel 1's in T' .

For each $i \leq d$, let c_i denote the member of $\max(C)$ extending s_i . Let I_0 denote the set of indices $i \leq d$ such that the immediate successor of c_i in T is 0; let I_1 denote the set of indices $i \leq d$ such that the immediate successor of c_i in T is 1. Note that since c_d is a coding node, d must be in I_0 . Let

$$(24) \quad C^- = \{c_i : i \in I_0\} \cup \{\text{splitpred}_T(c_i) : i \in I_1\}.$$

For any $T'' \in [r_k(T'), T']$, define $X \in \text{Ext}_{T''}(A, C^-)$ if and only if

- (1) $X = \{x_i : i \leq d\}$ and for each $i \leq d$, $x_i \supseteq s_i$;
- (2) There is a coding node c in T'' such that for each $i \in I_0$, $|x_i| = |c|$ and the immediate successor of x_i is 0; and for each $i \in I_1$, $x_i = \text{splitpred}_{T''}(y_i)$ for some $y_i \supseteq s_i$ in T'' of length $|c|$ such that the immediate extension of y_i is 1.
- (3) The set $A \cup \{x_i : i \leq d\}$ satisfies the Parallel 1's Criterion.

(3) is equivalent to saying that $\{x_i : i \in I_0\}$ along with the rightmost paths extending $\{x_i : i \in I_1\}$ to the level of the coding node c forms a minimal pre-extension of A to a copy of C .

As in Theorem 22, without loss of generality, and in order to simplify notation, assume that C contains a member of $0^{<\omega}$; let i_0 denote the integer such that s_{i_0} is the node of A_e which is all 0's and has the same length as the other nodes in A_e . Let L denote the collection of all $l < \omega$ such that there is a member of $\text{Ext}_{T'}(A, C^-)$ with maximal nodes of length l .

For each $i \in (d+1) \setminus \{i_0\}$, let $T'_i = \{t \in T' : t \supseteq s_i\}$. Let $T'_{i_0} = \{t \in T' \cap 0^{<\omega} : t \supseteq s_{i_0}\}$, the collection of all leftmost nodes in T' extending s_{i_0} . Let $\kappa = \beth_{2d}$. The following forcing notion \mathbb{P} will add κ many paths through each T'_i , $i \in (d+1) \setminus \{i_0\}$, and one path through T'_{i_0} . The present case is handled similarly to Case (a) of Theorem 22, so much of the current proof refers back to the proof of Theorem 22.

Let \mathbb{P} be the set of conditions p such that p is a function of the form

$$p : (d+1) \times \vec{\delta}_p \rightarrow T',$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, and there is some coding node $c_{n(p)}^{T'}$ such that

- (i) $l_p := |c_{n(p)}^{T'}|$;
- (ii) If $i \leq d$ and $\delta \in \vec{\delta}_p$, then $|c_{n(p)-1}^{T'}| < |p(i, \delta)| \leq l_p$ and $p(i, \delta) \in T'_i$.
 - (α) If $i \in I_1$, then $p(i, \delta) = \text{splitpred}_{T'}(y)$ for some $y \in T'_i \restriction l_p$ which has immediate extension 1.
 - (β) If $i \in I_0$, then $p(i, \delta) \in T'_i \restriction l_p$ and has immediate extension 0.

The partial ordering on \mathbb{P} defined as follows: $q \leq p$ if and only if either

- (1) $l_q = l_p$ and $q \supseteq p$ (so also $\vec{\delta}_q \supseteq \vec{\delta}_p$); or else
- (2) $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and
 - (i) For each $\delta \in \vec{\delta}_p$ and $i \leq d$, $q(i, \delta) \supset p(i, \delta)$;

- (ii) Whenever $(\alpha_0, \dots, \alpha_d)$ is a strictly increasing sequence in $(\vec{\delta}_p)^{d+1}$ and $\{p(i, \alpha_i) : i \leq d\} \in \text{Ext}_{T'}(A, C^-)$, then also $\{q(i, \alpha_i) : i \leq d\} \in \text{Ext}_{T'}(A, C^-)$.

Remark. We point out that whenever $\{p(i, \alpha_i) : i \leq d\} \in \text{Ext}_{T'}(A, C^-)$, then the set $\{p(i, \alpha_i) : i \in I_0\}$ along with the rightmost extensions of $p(i, \alpha_i)$, for all $i \in I_1$, to the length of the next coding node in T' forms a minimal pre-extension of a copy of C . On the other hand, if W is the set of $\{p(i, \alpha_i) : i \in I_0\}$ along with extensions of the members of $\{p(i, \alpha_i) : i \in I_1\}$ to the same level, where some but not all are extended along their rightmost extensions, and if W does not satisfy the Parallel 1's Criterion, then W is not a minimal pre-extension. Hence, we will not homogenize over copies of C extending such W , since such copies will not have the Strict Parallel 1's Criterion in any subtree of T' containing W .

Claims 1 and 2 in the proof of Theorem 22 show that (\mathbb{P}, \leq) is a partial ordering, and for each $p \in \mathbb{P}$ and each $n > l_p$, there is a $q \in \mathbb{P}$ with $l_q > n$ satisfying $q < p$. Claims 1 and 2 have the same proofs as before.

Let \mathcal{U} be a \mathbb{P} -name for a non-principal ultrafilter on L . For each $i \leq d$ and $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ be a \mathbb{P} -name for the α -th generic branch through T_i ; that is, $\dot{b}_{i,\alpha} = \{\langle p(i, \alpha), p \rangle : p \in \mathbb{P} \text{ and } \alpha \in \vec{\delta}_p\}$. For $i \in I_0$, for any condition $p \in \mathbb{P}$ and $\alpha \in \vec{\delta}_p$, p forces that $\dot{b}_{i,\alpha} \upharpoonright l_p = p(i, \alpha)$. For $i \in I_1$, for any condition $p \in \mathbb{P}$ and $\alpha \in \vec{\delta}_p$, p forces that $\text{splitpred}_{T'}(\dot{b}_{i,\alpha} \upharpoonright l_p) = p(i, \alpha)$. We write sets $\{\alpha_i : i \leq d\}$ in $[\kappa]^{d+1}$ as vectors $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_d \rangle$ in strictly increasing order. For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_d \rangle \in [\kappa]^{d+1}$, rather than writing out $\langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d,\alpha_d} \rangle$ each time we wish to refer to these generic branches, we shall simply

$$(25) \quad \text{let } \dot{b}_{\vec{\alpha}} \text{ denote } \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d,\alpha_d} \rangle.$$

For $l \in L$, we shall write $\dot{b}_{\vec{\alpha}} \upharpoonright l$ to mean $\{\dot{b}_{i,\alpha_i} \upharpoonright l : i \in I_0\} \cup \{\text{splitpred}_{T'}(\dot{b}_{i,\alpha_i} \upharpoonright l) : i \in I_1\}$.

The coloring c on $\text{Ext}_{T'}(A, C)$ induces a coloring c' on $\text{Ext}_{T'}(A, C^-)$ as follows. Since T' is end-homogeneous for c , for each minimal pre-extension X in T' , c is constant on the set of $Z \in \text{Ext}_{T'}(A, C)$ extending X . Define $c'(X)$ to be the color of c on all $Z \in \text{Ext}_{T'}(A, C)$ extending X . Given any $Y \in \text{Ext}_{T'}(A, C^-)$ which is not a minimal pre-extension, there is a minimal pre-extension X such that Y extends X . Define $c'(Y) = c'(X)$. By end-homogeneity of T' , each $Z \in \text{Ext}_{T'}(A, C)$ extending Y has c -color equal to $c'(X)$, and hence also $c'(Y)$. In particular, for each $Z \in \text{Ext}_{T'}(A, C)$ $c'(Z^-) = c(Z)$, where $Z^- = \{z_i : i \in I_0\} \cup \{\text{splitpred}_{T'}(z_i) : i \in I_1\}$.

Extend each node in A_e leftmost to its extension in $\max(B)$, and call this A' . Since A is thin in B , A' adds no new parallel 1's so it is an extension of A which satisfies the Parallel 1's Criterion and is strictly similar to A . By almost exactly the same proof as Section 2 in the proof of Theorem 22, there are infinite pairwise disjoint sets $K'_i \subseteq \kappa$, $i \leq d$, and conditions $p_{\vec{\alpha}}$, $\vec{\alpha} \in \prod_{i \leq d} K'_i$, such that these conditions are pairwise compatible, have the same images in T , and force the same color for $c'(\dot{b}_{\vec{\alpha}} \upharpoonright l)$ for ultrafilter many levels l . Moreover, there are nodes $\{t_i^* : i \leq d\}$ which extend A' , hence also extend $\{s_i : i \leq d\}$, and $\{t_i^* : i \leq d\}$ is a member of $\text{Ext}_{T'}(A, C^-)$. For $i \in I_0$, let $u_i^* = t_i^*$. For $i \in I_1$, let u_i^* be the leftmost extension of t_i^* in $T' \upharpoonright l^*$. Let l^* denote the length of the nodes u_i^* .

Extend each node u in $B \setminus A'$ to its leftmost extension in $T' \upharpoonright l^*$ and label that extension u^* . Let

$$(26) \quad U^* = \{u_i^* : i \leq d\} \cup \{u^* : u \in \max(B) \setminus A'\}.$$

Since $\max(B)$ contains a coding node and each u in $\max(B)$ is extended leftmost, and since $\{u_i^* : i \leq d\}$ satisfies the Parallel 1's Criterion over A , all parallel 1's in U^* are already witnessed in B . Hence, U^* satisfies the Parallel 1's Criterion. It follows that U^* can be extended in T' to a member of $r_{k+1}[B, T']$. Most importantly, any extension of the nodes in $\{u_i^* : i \leq d\}$ to a member of $\text{Ext}_{T'}(A, C^-)$ can be extended via the forcing to obtain another member of $\text{Ext}_{T'}(A, C^-)$ with \mathcal{C}' -color ε^* . The members of U^* will not be a level of the strong coding tree S we are constructing, but the maximal members of $r_{k+1}(S)$ will extend the maximal members of U^* .

For each n such that there is a coding node in $\max(r_n(T'))$, let n^- denote the maximal integer less than n such that there is a coding node in $\max(r_{n^-}(T'))$. Thus, for $s \in \max(r_{n^-}(T'))$ and $t \in \max(r_n(T'))$, $(|s|, |t|]$ is an interval of T' . Let $(n_j)_{j < \omega}$ denote the indices such that for any member V of $\mathcal{AR}_{n_j}|T'$ extending U^* , there is at least one member of $\text{Ext}_V(A, C^-)$ between the levels of $r_{n_j^-}(V)$ and V . Note that $n_0^- \geq k$, since the least member of $\mathcal{AR}_{n_j}|T'$ extending U^* must include splitting predecessors or a node of the same length as some coding node above the coding node in $\max(B)$.

The following Induction Hypothesis is vacuously true for $j = 0$.

Induction Hypothesis. Suppose that we have chosen $r_{n_{j-1}}(S)$ so that every member of $\text{Ext}_{r_{n_{j-1}}(S)}(A, C^-)$ is colored ε^* by f' .

If $j \geq 1$, take V_j to be any member of $r_{n_j}[r_{n_{j-1}}(S), T']$. If $j = 0$, take V_0 to be any member of $r_{n_0}[B, T']$ such that the members of $\max(r_{k+1}(V_0))$ extend the members of U^* . Let e denote the index of the maximal coding node in V_j , so that $c_e^{V_j}$ is the coding node in $\max(V_j)$. Let l_e denote the length of $c_e^{V_j}$. Let V_{left} denote the set of all nodes in $\max(V_j)$ which do not split between the levels of $|c_{e-1}^{V_j}|$ and l_e . Let V_{split} denote the set of all splitting nodes in V_j between the levels of $|c_{e-1}^{V_j}|$ and l_e . For each $v \in V_{\text{split}}$, let v_0 denote the extension of $v \cap 0$ in V_j to level l_e , and let v_1 denote the extension of $v \cap 1$ in V_j to level l_e . Note that

$$(27) \quad V_j = V_{\text{left}} \cup \{v_i : v \in V_{\text{split}}, i < 2\}.$$

We point out that for $v \in V_{\text{left}}$, the immediate successor of v in T is 0; for $v \in V_{\text{split}}$, the immediate successor of v_i in T is i , for $i = 0, 1$.

Recall that I_1 denotes the set of those $i \leq d$ for which t_i^* is a splitting node, and I_0 denotes the set of those $i \leq d$ for which t_i^* is not a splitting node. For $i \in I_1$, let X_i denote the set $V_{\text{split}} \cap T_i$; for $i \in I_0$, let X_i denote the set $V_{\text{left}} \cap T_i$. Note that X_i is exactly the set of all nodes in V_j which extend t_i^* and are in some member of $\text{Ext}_{V_j}(A, C^-)$. For each $i \leq d$, let J_i be a subset of K'_i with the same size as X_i . For each $i \leq d$, label the members of X_i as $q(i, \alpha_i)$, where $\alpha_i \in J_i$. Let \vec{J} denote the set of those $\langle \alpha_0, \dots, \alpha_d \rangle \in \prod_{i \leq d} J_i$ such that the set $\{q(i, \alpha_i) : i \leq d\}$ is in $\text{Ext}_{T'}(A, C^-)$. Then for each $\vec{\alpha} \in \vec{J}$ and all $i \leq d$,

$$(28) \quad q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i).$$

Construct a condition $q \in \mathbb{P}$ such that $q \leq p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \vec{J}$ as follows: Let $\vec{\delta}_q = \bigcup \{ \vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J} \}$. For each pair (i, γ) such that $i \leq d$ and $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $j < k^*$ such that $\gamma = \delta_{\vec{\alpha}}(j)$. As in Case (a), for any other $\vec{\beta} \in \vec{J}$ for which $\gamma \in \vec{\delta}_{\vec{\beta}}$, it follows that $p_{\vec{\beta}}(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t_{i,j}^*$ and $\vec{\delta}_{\vec{\beta}}(j) = \gamma$. Furthermore, this j is not equal to j_i , so $t_{i,j}^*$ is not a member of V . If $i \in I_0$, let $q(i, \gamma)$ be the leftmost extension of $t_{i,j}^*$ in $T \upharpoonright l_e$. If $i \in I_1$, let $q(i, \gamma)$ be the leftmost extension of $t_{i,j}^*$ to a splitting node in T between the levels of $|c_e^{V_j}|$ and l_e . Such a splitting node must exist, since $i \in I_1$ implies that $t_{i,j}^*$ is a splitting node, which implies $t_{i,j}^*$ and t_d^* have no parallel 1's. Hence, the leftmost extension x of $t_{i,j}^*$ in T to level l_e must have no parallel 1's with $q(d, \alpha_d)$, and it follows that $\text{splitpred}_T(x)$ is between levels $|d_{e-1}^{V_j}|$ and l_e . Therefore, $q(i, \gamma)$ is well-defined. Define

$$(29) \quad q = \bigcup_{i \leq d} \{ \langle (i, \alpha), q(i, \alpha) \rangle : \alpha \in \vec{\delta}_q \}.$$

By a proof similar to that of Claim 8, it follows that $q \leq p_{\vec{\alpha}}$, for each $\vec{\alpha} \in \vec{J}$.

Take an $r \leq q$ in \mathbb{P} which decides some l_j in L satisfying the following: l_j is strictly greater than the length of the next coding node above the maximal coding node in V_j , and for all $\vec{\alpha} \in \vec{J}$, $c'(\vec{b}_{\vec{\alpha}} \upharpoonright l_j) = \varepsilon^*$. Without loss of generality, we may assume that the maximal nodes in r have length l_j .

Let $n(r)$ denote the index such that $c_{n(r)}^{T'}$ is the coding node in T' with length equal to $|r(i, \alpha_i)|$ for any $i \in I_0$. Then, for $\gamma \in \vec{\delta}_q$, for all $i \in I_0$, $r(i, \gamma)$ has length l_j , and for all $i \in I_1$, $r(i, \gamma)$ has length strictly between $|c_{n(r)-1}^{T'}|$ and l_j .

We now construct $r_{n_j}(S)$ extending the nodes in $V_{\text{left}} \cup V_{\text{split}}$ and which we shall show is a member of $r_{n_j}[r_{n_j-1}(S), T']$.

Let c_{V_j} denote the coding node in $\max(V_j)$. If c_{V_j} is in X_i for some $i \in I_0$, then for the $\alpha_i \in J_i$ such that $c_{V_j} = q(i, \alpha_i)$, let c^- denote $r(i, \alpha_i)$ and let m be such that $c_m^{T'}$ is a coding node in T' extending c^- at least two intervals above $r(i, \alpha_i)$. If c_{V_j} is not in X_i for any $i \in I_0$, then extend c_{V_j} leftmost to the (coding node) level of the longest nodes in r can label this node c^- ; then extend c^- at least two more intervals longer to a coding node $c_m^{T'}$. Let $l_* = |c_m^{T'}|$. This extra length will allow for an application of the Passing Number Choice Lemma.

For each $v \in V_{\text{left}} \setminus (\bigcup_{i \in I_0} X_i)$, let v_* be the leftmost extension of v in T' of length l_* . For each $v \in V_{\text{split}} \setminus (\bigcup_{i \in I_1} X_i)$, extend both v_0 and v_1 along their leftmost extensions to level $|c_{m-1}^{T'}|$; call these v'_0 and v'_1 . Since v has no parallel 1's with c_{V_j} , v'_0 and v'_1 have no parallel 1's with $c_m^{T'} \upharpoonright |c_{m-1}^{T'}|$. Thus, both v'_0 and v'_1 extend to splitting nodes in the interval between $c_{m-1}^{T'}$ and $c_m^{T'}$. Take v_0^* to be the leftmost extension of v'_0 to length l_* and take v_1^* to be the rightmost extension of v'_1 to length l_* . Then the passing number of v_0^* at $c_m^{T'}$ is 0 and the passing number of v_1^* at $c_m^{T'}$ is 1. Moreover, any parallel 1's between either of these nodes and any extension of any other node in V_j below level l_* is witnessed in V_j .

For each $v \in X_i$, $i \leq d$, $v = q(i, \alpha_i)$ for some $\alpha_i \in J_i$. For $i \in I_0$, extend v to $r(i, \alpha_i)$ and then let v^* be the leftmost extension of $r(i, \alpha_i)$ to length l_* . For $i \in I_1$, $r(i, \alpha_i)$ is a splitting node extending v . Take both the left and right extensions of $r(i, \alpha_i)$ to the length of $|c^-|$; this is the length of the nodes $r(i, \alpha)$ for $i \in I_0$. Call these v_0^- and v_1^- . These both have no parallel 1's in common with c^- , so their leftmost extensions to level $|c_{m-1}^{T'}|$ also have no

parallel 1's with $c_m^{T'} \restriction |c_{m-1}^{T'}|$. Call these v_0^+ and v_1^+ . Thus in this interval, both v_0^+ and v_1^+ split. Take v_0^* to be the leftmost extension of v_0^+ and v_1^* to be the rightmost extension of v_1^+ of length l_* .

Let

$$\begin{aligned}
N_j = & \{v_* : v \in V_{\text{left}} \setminus (\bigcup_{i \in I_0} X_i)\} \\
& \cup \{v_a^* : v \in V_{\text{split}} \setminus (\bigcup_{i \in I_1} X_i), a \in \{0, 1\}\} \\
& \cup \{v^* : v \in X_i, i \in I_0\} \\
(30) \quad & \cup \{v_a^* : v \in X_i, i \in I_1, a \in \{0, 1\}\}.
\end{aligned}$$

By a similar argument as in the proof of Theorem 22, $r_{n_j-1}(S) \cup N$ satisfies the Parallel 1's Criterion. Finally, apply the Passing Number Choice Lemma 23 to extend each of these nodes in N to another level set M_j so that $(r_{n_j-1} \cup M_j) \stackrel{s}{\sim} r_{n_j}(T)$. Define $r_{n_j}(S) = r_{n_j-1} \cup M_j$. Then $r_{n_j}(S)$ truly is a member of $r_{n_j}[r_{n_j-1}(S), T']$, and every $Y \in \text{Ext}_{r_{n_j}(S)}(A, C^-)$ satisfies $c'(Y) = \varepsilon^*$.

Let $S = \bigcup_{j < \omega} r_{n_j}(S)$. Then $S \in [B, T']$ and satisfies the theorem. \square

We point out that the tree S constructed in Lemma 34 is not necessarily regular; that is, splitting nodes in a given interval of S are not necessarily increasing in length in terms of reverse lexicographic order. We do not require strong coding trees to be regular. However, if one wishes, one may take a strong coding subtree of S which is regular.

Lemma 35. *Let T be a strong coding tree, and let A, C, c be as in Lemma 32, where $\max(C)$ contains a coding node. Then there is a strong coding subtree $S \leq T$ such that for each copy A' of A in S , c is homogeneous on $\text{Ext}_S^{SP}(A', C)$.*

Proof. Let $(k_i)_{i < \omega}$ be the sequence of integers such that $r_{k_i}(T)$ contains a copy of A which is thin in $r_{k_i}(T)$. Fix $U_0 = r_{k_0}(T)$, and let S_{-1} denote T .

Claim 10. *For $i < \omega$, given $U_i \in \mathcal{AT}_{k_i}(S_{i-1})$, there is a $S_i \in [U_i, S_{i-1}]$ such that for each copy A' of A thin in U_i , $\text{Ext}_{S_i}^{SP}(A', C)$ is homogeneous for c .*

Proof. Enumerate all copies of A which are thin in U_i as $\langle A_0, \dots, A_n \rangle$. Since these are thin in U_i and are to be extended to copies of C which are thin in the tree S_i we are constructing, we may, without loss of generality, assume that the maximal level set of each A_j is contained in $\max(U_i)$. Apply Lemma 32 to obtain a $T_0 \in [U_i, S_{i-1}]$ which is end-homogeneous for $\text{Ext}_{T_0}^{SP}(A_0, C)$. Then apply Lemma 34 to obtain $T'_0 \in [U_i, T_0]$ such that $\text{Ext}_{T'_0}^{SP}(A_0, C)$ is homogeneous for c . Given T_j for $j < n$, apply Lemma 32 to obtain a $T_{j+1} \in [U_i, T'_j]$ which is end-homogeneous for $\text{Ext}_{T_{j+1}}^{SP}(A_{j+1}, C)$. Then apply Lemma 34 to obtain $T'_{j+1} \in [U_i, T_{j+1}]$ such that $\text{Ext}_{T'_{j+1}}^{SP}(A_{j+1}, C)$ is homogeneous for c . Let $S_i = T'_n$. \square

Given S_i satisfying Claim 10, let $U_{i+1} = r_{k_{i+1}}(S_i)$ and apply Claim 10 again to obtain S_{i+1} and let $U_{i+2} = r_{k_{i+2}}(S_{i+1})$. Define $U = \bigcup_{i < \omega} U_i$. Then for each thin copy A' of A in S , there is some i such that A' is contained in U_i ; and hence, c is homogeneous on $\text{Ext}_U^{SP}(A', C)$. Lastly, let S be a strong coding tree which is thin in U . Then each copy A' of A in S is thin in U , so $\text{Ext}_S^{SP}(A', C)$ is homogeneous for c . \square

A similar lemma holds for the setting of Case (a) in Theorem 22.

Lemma 36. *Let T be a strong coding tree and let A, C, c be as in Case (a) of Theorem 22; that is, the maximal level of C contains a splitting node. Then there is a strong coding tree $S \leq T$ such that for each copy A' of A in S , $\text{Ext}_S(A', C)$ is homogeneous for c .*

Proof. Similarly to the proof of Claim 10, by a fusion argument applying Case (a) of Theorem 22, we build a strong coding tree $S \leq T$ such that for each copy A' of A in S , $\text{Ext}_S(A', C)$ is homogeneous for c . \square

Remark. For the case that $\max(C)$ has a splitting node, we do not need to assume that the copy of A is thin in U ; for if a copy of A in S has no extensions to a copy of C , then the previous lemma vacuously holds.

Proof of Theorem 29. Let T be a strong coding tree and let Z be a finite subtree of T satisfying the Parallel 1's Criterion. Let c be a coloring of all the strongly similar copies of Z in T . Enumerate in increasing order of length the set of all the coding and splitting nodes of Z as $\langle z_i : i \leq n \rangle$. Let $l_i = |z_i|$ for each $i \leq n$, and let $l_{-1} = 0$. For each $i \leq n$, let A_i denote $\{t \upharpoonright l_j : t \in Z \text{ and } j < i\}$, and let $C_i = \{t \upharpoonright l_i : t \in Z \text{ and } j \leq i\}$. We apply a reverse induction argument using Lemmas 35 and 36 to prove the theorem.

For each copy A'_n of A_n in T , define $c_{A'_n}$ on $\text{Ext}_T(A'_n, C_n)$ by $c_{A'_n}(X) = c(A'_n \cup X)$, for each $X \in \text{Ext}_T(A'_n, C_n)$. If z_n is a coding node apply Lemma 35, and if z_n is a splitting node apply Lemma 36, to obtain $S_n \leq T$ such that for each copy A'_n of A_n in S_n , $\text{Ext}_{S_n}^{SP}(A'_n, C_n)$ is homogeneous for the coloring $c_{A'_n}$ on $\text{Ext}_{S_n}^{SP}(A'_n, C_n)$. This induces a coloring c^{n-1} on all the copies of A_{n-1} in S_n : For each V a copy of A_{n-1} in S_n , define $c^{n-1}(V) = c(V \cup X)$ for any/every $X \in \text{Ext}_{S_n}^{SP}(V, C_n)$. This is well-defined, since c is homogeneous on $\text{Ext}_{S_n}^{SP}(V, C_n)$.

Continue the reverse induction as follows: Given $i < n$ and the coloring c^i on all copies of A_i in S_{i+1} , apply Lemma 35 if C_i has a coding node, and apply Lemma 36 if C_i has a splitting node to obtain a strong coding tree $S_i \leq S_{i+1}$ such that for each copy A'_i of A_i in S_i , c^i is homogeneous on $\text{Ext}_{S_i}^{SP}(A'_i, C_i)$.

At the end of the reverse induction, we obtain a strong coding tree $S_0 \leq S_1$ such that the stem of each copy of Z in S_0 has the same induced c^0 color. Let $S = S_0$. This implies that c takes the same color on each copy Y of Z in S such that Y satisfies the Strict Parallel 1's Criterion in S . \square

6. INCREMENTAL STRONG CODING TREES AND SETS OF WITNESSING CODING NODES

This section develops the notion of incremental strong coding trees, and two lemmas which will be applied in the next section aid the proof of the main Ramsey theorem of this paper. The lemmas will be applied to construct a special kind of strong coding tree $S \subseteq T$ and a collection of witnessing coding nodes $W \subseteq T$ so that at each level of a coding node c_n^S of S , each new set of parallel 1's in S at level $|c_n^S|$ is witnessed by a member of W which has length in the interval between the longest splitting node in S and $|c_n^S|$.

Given any strong coding tree T , let d_k^T denote the k -th critical node of T so that $\langle d_k^T : k < \omega \rangle$ is an enumeration of the set of all coding and splitting nodes in T , in order of increasing length. Letting c_n^T denote the n -th coding node of T , let k_n denote the index such that $c_n^T = d_{k_n}^T$.

Lemma 37. *Given any strong coding tree T , there is a strong coding tree $S \subseteq T$ isomorphic to T and a collection of witnessing coding nodes $W = \{w_n : n < \omega\} \subseteq T$ such that the following hold. For each $n < \omega$,*

- (1) w_n is a coding node in T which has no parallel 1's with any member of S .
- (2) $|d_{k_n-1}^S| < |w_n^\wedge| < |w_n| < |c_n^S|$.
- (3) Letting l_{k_n} denote $|c_n^S|$, for each node $t \in S \upharpoonright (l_{k_n} + 1)$, $t(l_{k_n}) = 1$ if and only if $t(|w_n|) = 1$.

Proof. We construct a subtree S of T which is isomorphic to T along with a collection W of coding nodes from T with properties (1) - (3). Let $r_{k_0}(S) = r_{k_0}(T)$. This is the subtree of T containing all splitting nodes of T occurring before the least coding node in T , $c_0^T = d_{k_0}^T$.

We now construct w_0 and $r_{k_0+1}(S)$. Let $\{s_p^0 : p \leq q_0\}$ denote the maximal nodes in $T \upharpoonright (|c_0^T| + 1)$, where we let $s_{q_0}^0$ denote the immediate extension of c_0^T in T . Let w_0^\wedge denote the least splitting node in $T \cap 0^{<\omega}$ such that $|w_0^\wedge| > |c_0^T|$. Extend w_0^\wedge to a coding node $w_0 = c_{m_0}^T$ for some $m_0 \geq 2$. Extend each node s_p^0 , $p \leq q_0$, via their leftmost extensions in T to length $|c_{m_0-1}^T| + 1$; label these nodes t_p^0 . Since each t_p^0 has no parallel 1's with $w_0 \upharpoonright (|c_{m_0-1}^T| + 1)$, each t_p^0 splits between the levels of $|c_{m_0-1}^T|$ and $|w_0|$. For each $p < q_0$ for which $s_p^0(|c_0^T|) = 0$, take the leftmost branch of t_p^0 in T of length $|w_0| + 1$ and call it u_p^0 . For each $p < q_0$ for which $s_p^0(|c_0^T|) = 1$, take the rightmost extension of t_p^0 in T of length $|w_0| + 1$ and call it u_p^0 . Note the following:

- (1)₀ w_0 is a coding node in T , and w_0 and u_p^0 have no parallel 1's, for all $p \leq q_0$.
- (3)₀ For each $p \leq q_0$, $u_p^0(|w_0|) = s_p^0(|c_0^T|)$.

(1)₀ holds because w_0 and u_p^0 have no parallel 1's up to the length of $c_{m_0-1}^T$; and between that level and the level of w_0 , all nodes u_p^0 which take leftmost paths have no parallel 1's with w_0 in that interval, and all nodes u_p^0 which take rightmost paths in that interval, if they are to go right at the level of w_0 must have no parallel 1's with the coding node w_0 . So no parallel 1's were introduced in this final interval.

Next, take $u_{q_0}^0$, which is the extension of c_0^T , and extend it to a coding node $c_{n_0}^T$ in T , where $n_0 \geq m_0 + 2$. Extend each node u_p^0 , $p < q_0$, by the leftmost extension in T to length $|c_{n_0-1}^T| + 1$, and label this extension v_p^0 . For each $p < q_0$ for which $s_p^0(|c_0^T|) = 0$, extend v_p^0 along the leftmost extension in T to length $|c_{n_0}^T|$ and label this x_p^0 . Note that x_p^0 is immediately extended by 0 in S . For each $p < q_0$ for which $s_p^0(|c_0^T|) = 1$, note that v_p^0 and $c_{n_0}^T \upharpoonright (|c_{n_0-1}^T| + 1)$ have no parallel 1's. It follows from Fact ?? that each such v_p^0 splits between the levels of $|c_{n_0-1}^T|$ and $|c_{n_0}^T|$. Extend each of these v_p^0 along the rightmost extension in T to length $|c_{n_0}^T|$ and label this x_p^0 . Note that each of these x_p^0 is immediately extended by 1 in T .

Finally, let

$$(31) \quad r_{k_0+1}(S) = r_{k_0}(T) \cup \{x_p^0 : p < q_0\} \cup \{c_{n_0}^T\}.$$

Since $d_{k_0-1}^S = d_{k_0-1}^T$, $w_0 = c_{m_0}^T$, and $c_0^S = c_{n_0}^T$, where $m_0 < n_0$, we have

$$(2)_0 \quad |d_{k_0-1}^S| < |w_n^\wedge| < |w_n| = |c_{m_0}^T| < |c_{n_0}^T| = |c_0^S|.$$

For each node $y \in S \upharpoonright (l_{k_0} + 1)$, where $l_{k_0} = |c_0^S|$, we have that $y = x_p^0 \frown^{i_p}$ for some $p < q_0$ and some $i_p < 2$ such that $s_p^0(|c_0^T|) = i_p$. Thus, $i_p = 1$ if and only if $s_p^0(|c_0^T|) = 1$ if and only if $u_p^0(|w_0|) = 1$. Thus,

- (3)₀ For each node $t \in S \upharpoonright (l_{k_0} + 1)$, $t(l_{k_0}) = 1$ if and only if $t(|w_0|) = 1$ if and only if $t(|c_0^T|) = 1$.

Thus, (1) - (3) hold for the construction of S up to its first coding node. Moreover, $r_{k_0+1}(S)$ is isomorphic to $r_{k_0+1}(T)$.

Now, suppose we have constructed S up to its i -th coding node c_i^S in such a way that $r_{k_i+1}(S)$ is isomorphic to $r_{k_i+1}(T)$ and chosen w_j , $j \leq i$, satisfying (1) - (3). Let n_i be the index such that $c_i^S = c_{n_i}^T$. Let $\text{Spl}(T, i+1)$ denote the set of those $s \in T \upharpoonright (|c_i^T| + 1)$ such that s splits in T before the level of c_{i+1}^T .

Let a_j^T , $j < j_*$, denote the maximal nodes in $r_{k_i+1}(T)$. Abuse notation, and let $\text{Spl}(T, i+1)$ denote the set of those $j < j_*$ such that $a_j^T \in \text{Spl}(T, i+1)$. We may assume without loss of generality that our indexing satisfies the following: Whenever $j < j'$ are both in $\text{Spl}(T, i+1)$, then the least splitting node in T extending a_j^T has smaller length than the least splitting node in T extending $a_{j'}^T$. Let a_j^S , $j < j_i$, denote the maximal nodes in $r_{k_i+1}(S)$ so that the isomorphism between $r_{k_i+1}(T)$ and $r_{k_i+1}(S)$ maps each a_j^T to a_j^S .

Let j_0 be least in $\text{Spl}(T, i+1)$. Extend $a_{j_0}^S$ to the least splitting node in T extending it, call it $s_{j_0}^0$. Extend all other a_j^S , $j \in j_* \setminus \{j_0\}$, via their leftmost extensions to same length as $s_{j_0}^0$, and call these s_j^0 . Let

$$(32) \quad r_{k_i+2}(S) = r_{k_i+1}(S) \cup \{s_j^0 : j < j_*\},$$

and let $d_{k_i+1}^S = s_{j_0}^0$. Since none of the extensions have parallel 1's between $|c_i^S| + 2$ and $s_{j_0}^0$, $r_{k_i+2}(S)$ is isomorphic to $r_{k_i+2}(T)$.

Next, let j_1 be least in $\text{Spl}(T, i+1) \setminus \{j_0\}$. Extend $s_{j_1}^0$ to the least splitting node in T extending it, and label this $s_{j_1}^1$. Take $s_{j_0}^0 \hat{\ } 0$ and $s_{j_0}^0 \hat{\ } 1$ and extend them via their leftmost extensions in S to the same height as $s_{j_1}^1$ and label these extensions $s_{j_0,0}^1$ and $s_{j_0,1}^1$, respectively. For all $j \in j_* \setminus \{j_0, j_1\}$, extend s_j^0 via its leftmost extension in T to level of $s_{j_1}^1$, and label these s_j^1 . Let

$$(33) \quad r_{k_i+3}(S) = r_{k_i+2}(S) \cup \{s_{j_0,0}^1, s_{j_0,1}^1\} \cup \{s_j^1 : j \in j_* \setminus \{j_0\}\},$$

and let $d_{k_i+2}^S = s_{j_1}^1$. Again, $r_{k_i+3}(S)$ is isomorphic to $r_{k_i+3}(T)$, since none of the extensions between $|c_i^S| + 2$ and $s_{j_1}^1$ have parallel 1's.

In general, suppose $1 \leq m < |\text{Spl}(T, i+1)| - 1$ and we have chosen $s_{j_r,0}^m$, $s_{j_r,1}^m$, $r < m$, for the first m many members of $\in \text{Spl}(T, i+1)$. We have also chosen a splitting node $s_{j_m}^m$ in T , where j_m the m -th member of $\text{Spl}(T, i+1)$, and s_j^m , $j \in j_* \setminus \{j_r : r \leq m\}$, leftmost extensions of a_j^S of the same length as $s_{j_m}^m$. Let j_{m+1} be least in $\text{Spl}(T, i+1)$ above j_m . Then choose a splitting node in T extending $s_{j_{m+1}}^m$ and label this $s_{j_{m+1}}^{m+1}$. Extend each of $s_{j_r,0}^m$, $s_{j_r,1}^m$, $r < m$, as well as $s_{j_m}^m \hat{\ } 0$ and $s_{j_m}^m \hat{\ } 1$ via their leftmost extensions in T to the length of $s_{j_{m+1}}^{m+1}$, and label these $s_{j_r,0}^{m+1}$, $s_{j_r,1}^{m+1}$, for $r \leq m$. For each $j \in j_* \setminus \{j_r : r \leq m+1\}$, extend s_j^m via its leftmost extension in T to the length of $s_{j_{m+1}}^{m+1}$, and label this s_j^{m+1} . Let

$$(34) \quad r_{k_i+m+3}(S) = r_{k_i+m+2}(S) \cup \{s_{j_r,0}^{m+1}, s_{j_r,1}^{m+1} : r \leq m\} \cup \{s_j^{m+1} : j \in j_* \setminus \{j_r : r \leq m\}\},$$

and let $d_{k_i+m+2}^S = s_{j_{m+1}}^{m+1}$. Again, $r_{k_i+m+3}(S)$ is isomorphic to $r_{k_i+m+3}(T)$.

At the end of this construction, letting $m_* = |\text{Spl}(T, i+1)| - 1$, we have $s_{j_{m_*}}^{m_*}$ a splitting node extending $s_{j_{m_*}}^{m_*-1}$ which extends $a_{j_{m_*}}^S$; for all $m < m_*$, $s_{j_m,0}^{m_*}$ and $s_{j_m,1}^{m_*}$ the left and right

extensions of $a_{j_m}^S$; and for each $j \in j_* \setminus \text{Spl}(T, i+1)$, the leftmost extension s_j^{m*} of a_j^S . Furthermore, above the level of $r_{k_i}(S)$, none of these nodes have any parallel 1's.

Note that $k_{i+1} = (k_i + 1) + (m_* + 1)$. Let

$$(35) \quad r_{k_{i+1}}(S) = r_{k_i+m_*+1}(S) \cup \{s_{j_r,0}^{m*}, s_{j_r,1}^{m*} : r < m_*\} \cup \{s_j^{m*} : j \in j_* \setminus \{j_r : r < m_*\}\},$$

and let $d_{k_i+m_*+1}^S$, which is $d_{k_{i+1}-1}^S$, be $s_{j_{m_*}}^{m*}$. Then $r_{k_{i+1}}(S)$ is isomorphic to $r_{k_{i+1}}(T)$.

To finish the construction of $r_{k_{i+1}+1}(S)$, let $t_{j_{m_*},0}$ and $t_{j_{m_*},1}$ denote the immediate left and right extensions of $s_{j_{m_*}}^{m*}$ to the left and right in T , respectively. For all $m < m_*$ let $t_{j_m,0}$ and $t_{j_m,1}$ be the immediate extensions of $s_{j_m,0}^{m*}$ and $s_{j_m,1}^{m*}$ in T . For all $j \in j_* \setminus \text{Spl}(T, i+1)$, let t_j be the immediate extension in T of s_j^{m*} . Note that the set $\{t_{j,0}, t_{j,1} : j \in \text{Spl}(T, i+1)\} \cup \{t_j : j \in j_* \setminus \text{Spl}(T, i+1)\}$ has no parallel 1's between the levels of $|c_i^S| + 2$ and $|t_{m_*}|$. Relabel the nodes in $\{t_{j_m,0}, t_{j_m,1} : m \leq m_*\} \cup \{t_j : j \in j_* \setminus \text{Spl}(T, i+1)\}$ as s_p^{i+1} , $p \leq q_{i+1}$, letting $s_{q_{i+1}}^{i+1}$ denote the node which will be extended to the $(i+1)$ -st coding node in S .

We now construct w_{i+1} and $r_{k_{i+1}+1}(S)$. Let w_{i+1}^\wedge denote the least splitting node in $T \cap 0^{<\omega}$ such that $|w_{i+1}^\wedge| \geq |s_{q_{i+1}}^{i+1}| + 2$. Extend w_{i+1}^\wedge to a coding node $w_{i+1} = c_{m_{i+1}}^T$ for some m_{i+1} such that $|c_{m_{i+1}-2}^T| > |w_{i+1}^\wedge|$. Extend each node s_p^{i+1} , $p \leq q_{i+1}$, via their leftmost extensions in T to length $|c_{m_{i+1}-1}^T|$; label these nodes t_p^{i+1} . Since each t_p^{i+1} has no parallel 1's with $w_{i+1} \upharpoonright (|c_{m_{i+1}-1}^T| + 1)$, it follows that each t_p^{i+1} splits between the levels of $|c_{m_{i+1}-1}^T|$ and $|w_{i+1}|$.

For each $p < q_{i+1}$ for which s_p^{i+1} is either $t_{j_m,0}$ for some $m \leq m_*$ or else is t_j for some $j \in j_* \setminus \text{Spl}(T, i+1)$, take the leftmost branch of t_p^0 in T of length $|w_{i+1}| + 1$ and call it u_p^{i+1} . For each $p < q_{i+1}$ for which s_p^{i+1} extends $t_{j_m,1}$ for some $m \leq m_*$, take the rightmost extension of t_p^{i+1} in T of length $|w_{i+1}| + 1$ and call it u_p^{i+1} . Note the following:

- (1) _{$i+1$} w_{i+1} is a coding node in T , and for all $p \leq q_{i+1}$, w_{i+1} and u_p^{i+1} have no parallel 1's.
- (3) _{$i+1$} For each $p \leq q_{i+1}$, $u_p^{i+1}(|w_{i+1}|) = f(p)(|c_{m_{i+1}}^T|)$, where f is the strong similarity between $\{t_p^{i+1} : p \leq p_{i+1}\}$ and $r_{k_{i+1}}(T)^+$.

(1) _{$i+1$} holds because w_{i+1} and u_p^{i+1} have no parallel 1's up to the length of $c_{m_{i+1}-1}^T$; and between that level and the level of w_{i+1} , all nodes u_p^{i+1} which take leftmost paths have no parallel 1's with w_{i+1} in that interval, and all nodes u_p^{i+1} which take rightmost paths in that interval, if they are to go right at the level of w_{i+1} must have no parallel 1's with the coding node w_{i+1} . So no parallel 1's were introduced in this final interval.

Next, extend $u_{q_{i+1}}^{i+1}$ to a coding node $c_{n_{i+1}}^T$ in T , where $|c_{n_{i+1}-1}^T| > \text{lh}(w_{i+1})$. Label this c_{i+1}^S . Let $l_{i+1}^* = |c_{i+1}^S|$. Extend each node u_p^{i+1} , $p < q_{i+1}$, by the leftmost extension in T to the length of $c_{n_{i+1}-1}^T$, and label this extension v_p^{i+1} . For each $p < q_{i+1}$ for which $u_p^{i+1}(|w_{i+1}|) = 0$, extend v_p^0 along the leftmost extension in T to length l_{i+1}^* and label this x_p^{i+1} . Note that x_p^{i+1} is immediately extended by 0 in T . For each $p < q_{i+1}$ for which $u_p^{i+1}(|w_{i+1}|) = 1$, note that v_p^{i+1} and $c_{n_{i+1}}^T \upharpoonright (|c_{n_{i+1}-1}^T| + 1)$ have no parallel 1's. It follows that each such v_p^{i+1} splits between the levels of $|c_{n_{i+1}-1}^T|$ and $|c_{n_{i+1}}^T|$. Extend each of these v_p^{i+1} along the rightmost extension in T to length $|c_{n_{i+1}}^T|$ and label this x_p^{i+1} . Note that each of these x_p^{i+1} is immediately extended by 1 in T .

Finally, let

$$(36) \quad r_{k_{i+1}+1}(S) = r_{k_{i+1}}(S) \cup \{x_p^{i+1} : p < q_{i+1}\} \cup \{c_{n_{i+1}}^T\}.$$

One can easily check that we have

$$(2)_{i+1} \quad |d_{k_{i+1}-1}^S| < |w_{i+1}^\wedge| < |w_{i+1}| < |c_{i+1}^S|.$$

For each node $y \in S \upharpoonright (l_{k_{i+1}} + 1)$, where $l_{k_{i+1}} = |c_{i+1}^S|$, we have that $y = x_p^{i+1} \frown i_p$ for some $p < q_{i+1}$ and some $i_p < 2$ such that $s_p^{i+1}(|c_{i+1}^T|) = i_p$. Thus, $i_p = 1$ if and only if $s_p^{i+1}(|c_{i+1}^T|) = 1$ if and only if $u_p^{i+1}(|w_{i+1}|) = 1$. Thus,

$$(3)_{i+1} \text{ For each node } t \in S \upharpoonright (l_{k_{i+1}} + 1), t(l_{k_{i+1}}) = 1 \text{ if and only if } t(|w_{i+1}|) = 1.$$

Thus, (1) - (3) hold for the construction of S up to its first coding node. By construction, $r_{k_{i+1}+1}(S)$ is isomorphic to $r_{k_{i+1}+1}(T)$. \square

The second lemma of this section shows that given a strong coding tree T , there is a strong coding subtree $S \leq T$ such that for any finite subtree Z of S , for each minimal level l of a new set of parallel 1's in Z , if $|I_l| > 2$, then there is an $l' < l$ such that l' is a level of a new set of parallel 1's in Z satisfying $I_{l'} \subseteq I_l$ and $|I_{l'}| = |I_l| - 1$, where I_l was defined at the beginning of Section 7. Thus, all new sets of parallel 1's in S are incrementally increased in T . This will be used later to show that the number of strict envelopes of finite diagonal trees coding a fixed finite triangle-free graph G will yield the exact big Ramsey degree of G in the universal triangle-free graph \mathcal{H}_3 .

Definition 38 (Incrementally witnessed new parallel 1's). Let Z be a finite subtree of a strong coding tree T . Let c be a coding node in Z , $l = |c|$, and $\mathcal{O}_l = \{z \in Z \upharpoonright (l+1) : z(l) = 1\}$. Assume that $|\mathcal{O}_l| > 2$ and the parallel 1's in \mathcal{O}_l are not witnessed in Z below c .

We say that *the set of new parallel 1's in Z at c are incrementally witnessed in T* if the following hold. Let $p = |\mathcal{O}_l|$ and enumerate the members of \mathcal{O}_l as $\{t_i : i < p\}$. Let \mathcal{I} be the set of all proper subsets $I \subsetneq p$ such that $|I| \geq 2$ and the set of parallel 1's $\{t_i : i \in I\}$ at level l is not witnessed below c in Z . Let $q = |\mathcal{I}|$. There are levels l_j , $j < q$, such that letting s be the maximal splitting node of Z below c , we have $|s| < l_0 < l_1 < \dots < l_{q-1} < l$, and the following. Letting I_j denote $\{i < p : t_i(l_j) = 1\}$, then $\mathcal{O}_{l_j} = \{t_i \upharpoonright (l+1) : i \in I_j\}$.

- (1) For each $j < q$, l_j is a minimal level of parallel 1's in Z not witnessed in Z .
- (2) $j < k < q$ implies $|I_j| \leq |I_k|$.
- (3) For each $j < q$, there is a coding node w_j in T such that $l_j \leq |w_j| < l_{j+1}$, if $j+1 < q$; $l_{q-1} \leq |w_{q-1}| < l$; and $\{i < p : t_i(|w_j|) = 1\} = I_j$.
- (4) Moreover, we shall require these coding nodes $\{w_j : j < q\}$ in T to have the properties that $|s| < |w_0^\wedge| < |w_0| < |w_1^\wedge| < |w_1| < \dots < |w_{q-1}^\wedge| < |w_{q-1}| < l$, and the set $\{w_j : j < q\}$ has no parallel 1's, and has no parallel 1's with any node in Z .

If Z is a subtree (finite or infinite) of a strong coding tree T such that Z satisfies the Parallel 1's Criterion, we say that *all sets of new parallel 1's in Z are incrementally witnessed in T* if for each coding node c in Z , the set of new parallel 1's in Z at c are incrementally witnessed in T .

Remark. Whenever Z is a subtree of a strong coding tree T such that satisfies the Parallel 1's Criterion and all sets of new parallel 1's in Z are incrementally witnessed in T , it follows that Z along with the set of the witnessing coding nodes in T satisfying Definition 38 satisfies the Strict Parallel 1's Criterion as a subtree of T .

The second lemma of this section enhances the construction in Lemma 37. Instead of one witnessing coding node w_n in the interval between the the n -th coding node c_n^S of S and the maximal splitting node below c_n^S , we now add as many witnessing coding nodes as there are increments necessary to incrementally witness new parallel 1's at the level of c_n^S .

Lemma 39. *Let T be a strong coding tree. Then there is a strong coding tree $S \leq T$ such that all sets of new parallel 1's in S are incrementally witnessed in T .*

Proof. The proof proceeds by following the general construction for Lemma 37, changing the intermediate steps from adding one witnessing coding node to adding one witnessing coding node per increment, as we show here. It is important that at each stage of constructing the next level of S that the initial segment of S constructed so far be thin in T , so that at each level of the construction, we are free to choose which nodes extend to have parallel 1's.

Take $r_{k_0}(S)$ to be a thin subtree of $r_{k_0}(T)$. Since at the level of the least coding node in T there is only one node having 1 as its immediate extension, there are no parallel 1's in $r_{k_0+1}(T)$. Take a witnessing coding node w_0 from T and extend $r_{k_0}(S)$ to $r_{k_0+1}(S)$ exactly as in the proof of Lemma 37.

Now suppose that we have constructed $r_{k_n}(S)$ thin in T . Let $p_* = |\max(r_{k_n}(S))|$, and let $p < p_*$ be the number of nodes in $\max(r_{k_n+1}(T))$ which have passing number 1 at the coding node c_n^T . Let $\{s_i : i < p\}$ denote those nodes in $\max(r_{k_n}(S))$ whose extensions at the next coding node c_n^S in S must have passing number 1 so that $r_{k_n+1}(S)$ will be a strongly similar to $r_{k_n+1}(T)$. Let $\{s_i : p \leq i < p_*\}$ denote those nodes in $\max(r_{k_n}(S))$ whose extensions at the next coding node c_n^S in S must have passing number 0. Let \mathcal{I} denote the set of all proper subsets $I \subsetneq p$ with $|I| \geq 2$ such that the nodes $\{s_i : i \in I\}$ have no parallel 1's in $r_{k_n}(S)$; for those sets of nodes which have parallel 1's in $r_{k_n}(S)$ are already witnessed by some coding node in $r_{k_n}(S)$ as $r_{k_n}(S)$ satisfies the Parallel 1's Criterion. Enumerate \mathcal{I} as I_j , $j < q = |\mathcal{I}|$ in such a way that $j < k < q$ implies $|I_j| \leq |I_k|$.

Letting s_i^{-1} denote s_i , suppose $j < q$ and we have $\{s_i^{j-1} : i < p_*\}$. Take w_n^j to be a coding node in T such that the length of $(w_n^j)^\wedge$ is greater than the least coding node in T with length strictly greater than $|s_0^{j-1}|$. By the Passing Number Choice Lemma 23, we may extend all nodes in $\{s_i^{j-1} : i \in I_j\}$ leftmost until the length of the maximal coding node in T below w_n^j , at which point extend rightmost to have passing number 1 at w_n^j . Extend all nodes in $\{s_i^{j-1} : i \in p_* \setminus I_j\}$ leftmost to length of w_n^j . Denote these new nodes of length $|w_n^j|$ as $\{s_i^j : i < p_*\}$.

At the end of the q many steps, we have witnessing coding node w_n^{q-1} and nodes $\{s_i^{q-1} : i < p_*\}$ extending $\{s_i : i < p_*\}$. The nodes $\{s_i^{q-1} : i < p_*\}$ have no pre-determined parallel 1's among their possible extensions. Let I_q denote the set of those $i < p_*$ such that the nodes $\{s_i^{q-1} : i \in I_q\}$ need to extend to have passing number 1 at the n -th coding node of S in order to produce $r_{k_n+1}(S)$ strongly similar to $r_{k_n+1}(T)$. Let $i_* < p_*$ be the index of the node which needs to extend to the n -th coding node of S . Since the set $\{s_i^{q-1} : i < p_*\}$ has no predetermined parallel 1's, by the Passing Number Choice Lemma 23, there is a coding node c_n^S in T extending $s_{i_*}^{q-1}$ and nodes $\{s_i^q : i \in p_* \setminus \{i_*\}\}$ such that $r_{k_n}(S) \cup \{s_i^q : i \in p_* \setminus \{i_*\}\} \cup \{c_n^S\}$ is strongly similar to $r_{k_n+1}(T)$.

The construction establishes that the new parallel 1's in $r_{k_n+1}(S)$ are incrementally witnessed by the witnessing coding nodes w_n^j , $j < q$. Indeed, for each $j < q$, letting l_j be the

minimal level such that $\{i < p_* : s_i^q(l_j) = 1\} = I_j$, it follows from the construction that $|(w_n^j)^\wedge| < l_j \leq |w_n^j|$. This implies that Definition 38 is satisfied. \square

7. RAMSEY THEOREM FOR INCREMENTAL STRICT SIMILARITY TYPES

Theorem 29 in Section 5 is a Ramsey theorem for trees satisfying the Strict Parallel 1's Criterion inside a strong coding tree T . To obtain a more general Ramsey theorem not requiring the Strict Parallel 1's Criterion, we shall further thin to a tree $S \leq T$ of the form in the previous section so that all diagonal trees in S coding a given finite triangle-free graph will automatically satisfy the Strict Parallel 1's Criterion in T . Furthermore, in this section we present the notion of incremental strict similarity types. These will yield the actual finite big Ramsey degrees for finite triangle-free graphs in \mathcal{H}_3 .

Let T be a strong coding tree and let Z be a finite strongly diagonal subtree of T . Enumerate the maximal nodes in Z in order of increasing length as $\langle z_i : i < \tilde{i} \rangle$. For each $l < |z_{\tilde{i}-1}|$, let

$$(37) \quad I_l = \{i \leq \tilde{i} : |z_i| > l \text{ and } z_i(l) = 1\},$$

and let

$$(38) \quad \mathcal{O}_l = \{z_i \upharpoonright (l+1) : i \in I_l\}.$$

Thus, \mathcal{O}_l is the collection of all z_i which have passing number 1 at level l . Given l such that $|\mathcal{O}_l| \geq 2$, we say that the set of parallel 1's at level l is *witnessed by the coding node z_j in Z* if $z_i(|z_j|) = 1$ for each $i \in I_l$ and either $|z_j| \leq l$, or else $|z_j| > l$ and Z has no splitting nodes and no coding nodes of length in $[l, |z_j|]$.

A level $l < |z_{\tilde{i}-1}|$ is *the minimal level of a new set of parallel 1's in Z* if $|\mathcal{O}_l| \geq 2$ and for each $l' < l$ such that $|\mathcal{O}_{l'}| \geq 2$, there is at least one member of \mathcal{O}_l which is not an extension of any member of $\mathcal{O}_{l'}$; equivalently, $I_{l'} \not\supseteq I_l$. This definition holds whether or not the new set of parallel 1's is witnessed in Z .

Definition 40 (Admissible interval). If l is the minimal level of a new set of parallel 1's, the *admissible interval for I_l* is the interval $[l, l^*]$, where $l^* \geq l$ is maximal satisfying the following:

- (1) Z has no splitting node and no coding node of length in $[l, l^*]$.
- (2) For each $l' \in [l, l^*]$ such that $|\mathcal{O}_{l'}| \geq 2$ and $\mathcal{O}_{l'} \not\subseteq \mathcal{O}_l$, there is some $l'' < l$ such that $I_{l''} = I_{l'}$, and either $|\mathcal{O}_{l''}| \geq 2$ or else there is a coding node z_j with $|z_j| < l$ such that $z_i(|z_j|) = 1$ for all $i \in I_{l'}$.
- (3) For each $l' \in [l, l^*]$, $I_{l'}$ does not properly contain I_l .

For each $l' \in [l, l^*]$ such that $I_{l'} = I_l$, we also say that $[l, l^*]$ is the *admissible interval for $I_{l'}$* .

Let l be the minimal level of a new set of parallel 1's in Z . We say that l is the *minimal level of a new set of parallel 1's not witnessed in Z* if there is no $j < \tilde{i}$ satisfying $|z_j| \leq l^*$ such that $z_i(|z_j|) = 1$ for all $i \in I_l$; and if $l^* + 1$ is the length of some coding node z_j in Z , then $I_{l^*+1} \neq I_l$. Then for each l' in the admissible interval $[l, l^*]$ such that $I_{l'} = I_l$, we call l' a *level of a new set of parallel 1's not witnessed in Z* .

Definition 41 (Strict similarity sequence and type). Given Z a finite diagonal tree, list the minimal levels of new sets of parallel 1's in Z in increasing order as $l_0, \dots, l_{\tilde{j}-1}$. Enumerate

all nodes in Z^\wedge as $\langle u_m^Z : m < \tilde{m} \rangle$ in order of increasing length. Thus, each u_m^Z is either a splitting node or else is a coding node z_i for some $i < \tilde{i}$. The sequence

$$(39) \quad \langle \langle l_j : j < \tilde{j} \rangle, \langle I_{l_j} : j < \tilde{j} \rangle, \langle |u_m^Z| : m < \tilde{m} \rangle \rangle$$

is the *strict similarity sequence* of Z .

Let Y and Z be two finite strongly diagonal trees. Let

$$(40) \quad \langle \langle l_j^Y : j < \tilde{j}^Y \rangle, \langle I_{l_j^Y} : j < \tilde{j}^Y \rangle, \langle |u_m^Y| : m < \tilde{m}^Y \rangle \rangle$$

and

$$(41) \quad \langle \langle l_j : j < \tilde{j}^Z \rangle, \langle I_{l_j} : j < \tilde{j}^Z \rangle, \langle |u_m^Z| : m < \tilde{m}^Z \rangle \rangle$$

be their strict similarity sequences. We say that Y and Z have the same *strict similarity type* or are *strictly similar*, and write $Y \stackrel{ss}{\sim} Z$, if

- (1) Y and Z are strongly similar;
- (2) $\tilde{j}^Y = \tilde{j}^Z$ and $\tilde{m}^Y = \tilde{m}^Z$; thus, label these \tilde{j} and \tilde{m} , respectively;
- (3) For each $j < \tilde{j}$, $I_{l_j^Y} = I_{l_j^Z}$; and
- (4) The function $\varphi : \{l_j^Y : j < \tilde{j}\} \cup \{|u_m^Y| < \tilde{m}\} \rightarrow \{l_j^Z : j < \tilde{j}\} \cup \{|u_m^Z| < \tilde{m}\}$, defined by $\varphi(l_j^Y) = l_j^Z$ and $\varphi(u_m^Y) = u_m^Z$, is order preserving.

Let Z be a subtree of a strong coding tree T with strict similarity sequence $\langle \langle l_j^Z : j < \tilde{j} \rangle, \langle I_{l_j^Z} : j < \tilde{j} \rangle, \langle |u_m^Z| : m < \tilde{m} \rangle \rangle$ and such that the levels $\{l_j^Z : j < \tilde{j}\} \cup \{|u_m^Z| : m < \tilde{m}\}$, are all in different intervals of T . Such a Z is called a *representative for the strict similarity type of Z in T* . Define

$$(42) \quad \text{Sim}_T^{ss}(Z) = \{Y \subseteq T : Y \stackrel{ss}{\sim} Z \text{ and each level } l_j^Y, j < \tilde{j}, \text{ is in a different interval of } T\}.$$

The following notion of envelope is defined in terms of structure without regard to an ambient strong coding tree. In any given strong coding tree T , there will certainly be finite subtrees of T which have no envelope in T . This poses no problem to our intended application at the end of this section, which is to find a subtree S inside a given strong coding tree T such that each finite incremental strongly diagonal subtree of S has an envelope in T .

Definition 42 (Envelopes). Let Z be a finite incremental strongly diagonal tree. Let $\langle \langle l_j : j < \tilde{j} \rangle, \langle I_{l_j} : j < \tilde{j} \rangle, \langle |u_m| : m < \tilde{m} \rangle \rangle$ be the strict similarity sequence of Z . A finite tree $E(Z)$ is an *envelope* of Z if $E(Z) = Z \cup W$, where $W = \{w_j : j < \tilde{j}\}$ is a set of coding nodes satisfying the following: For each $j < \tilde{j}$,

- (1) w_j is in the admissible interval of l_j ; that is, $l_j \leq |w_j| \leq l_j^*$;
- (2) $I_{|w_j|} = I_{l_j}$;
- (3) w_j has no parallel 1's with any member of $Z \cup (W \setminus \{w_j\})$; and
- (4) $|w_j^\wedge| > l_{j-1}^*$ and there is no member of Z^\wedge with length in $[|w_j^\wedge|, |w_j|]$.

The set W is called the set of *witnessing coding nodes*, since they witness all parallel 1's not witnessed by a coding node in Z . Note that envelopes satisfy the Parallel 1's Criterion.

The next definition is closely tied to Definition 38, setting the stage for envelopes which have incrementally witnessed new parallel 1's.

Definition 43 (Incremental). Let Z be a finite diagonal subtree of a strong coding tree T . Let $l_j, j < \tilde{j}$, list in increasing order the minimal levels of new parallel 1's in Z . Then for each $j < \tilde{j}$, \mathcal{O}_{l_j} has size at least two. Let $I_j = \{i < \tilde{i} : z_i(l_j) = 1\}$. We say that Z is *incremental* if whenever $j < \tilde{j}$ and $|I_j| \geq 3$, then for each $I \subsetneq I_j$ with $|I| \geq 2$, there is a $k < j$ such that $I = I_k$.

Remark. We point out that given any finite incremental strongly diagonal tree Z , for any strong coding tree T , there is a stretched copy of Z in T which does have an envelope in T . Furthermore, whenever two strictly similar finite incremental strongly diagonal trees Y and Z have envelopes in T , then any envelope of Y is strongly similar to any envelope of Z .

Theorem 44 (Ramsey degrees for strongly diagonal trees). *Let T be a strong coding tree, thus coding the universal triangle-free graph \mathcal{H}_3 . Let G be a finite triangle-free graph, and let c be a coloring into finitely many colors of all finite strongly diagonal subtrees of T which code G . Let $n(G)$ be the number of different strict similarity types of incremental strongly diagonal trees coding G ; and let $\{Z_i : i < n(G)\}$ be a set of one representative from each of the different strict similarity types of finite incremental strongly diagonal trees coding G .*

Then there is a strong coding tree $S \leq T$ such that for each $i < n(G)$, each strictly similar copy of Z_i in S has the same c -color. Furthermore, each finite diagonal subtree of S coding G is strictly similar to Z_i for exactly one $i < n(G)$.

Proof. Let T, G , and $\{Z_i : i < n(G)\}$ be as in the hypotheses. Let E_i be the strict similarity type of an envelope of Z_i . Fix $i < n(G)$ and define c_i to be the coloring induced by c on all copies of E_i in T as follows: Given X a subtree of T such that $X \overset{ss}{\sim} E_i$, there is a unique subtree $Y \subseteq X$ such that $Y \overset{ss}{\sim} Z_i$ and X is an envelope of Y .

Apply Theorem 29 to obtain $T_0 \leq T$ such that c_0 has one color on all Strict Parallel 1's Criterion copies of E_0 in T_0 . Given $T_i, i < n(G) - 1$, apply Theorem 29 to obtain $T_{i+1} \leq T_i$ such that c_{i+1} has one color on all Strict Parallel 1's Criterion copies of E_{i+1} in T_{i+1} . Let T_* denote $T_{n(G)-1}$.

Next, obtain a strong coding tree $S \leq T_*$ and a set of witnessing coding nodes $W \subseteq T_*$ with the properties as in Lemma 39. Then S and W have the properties that for each finite strongly diagonal $A \subseteq S$ coding G , there is an $i < n(G)$ such that A is strictly similar to Z_i . Furthermore, there is an envelope $E(A)$ contained in T_* which is moreover Strict Parallel 1's Criterion in T_* . Thus, $c(A)$ is equal to the one color that c_i has on Strict Parallel 1's Criterion copies of $E(Z_i)$ in T_i . \square

8. THE UNIVERSAL TRIANGLE-FREE GRAPH HAS FINITE BIG RAMSEY DEGREES

In Theorem 44, we proved that given any finitary coloring of all copies of a given finite triangle-free graph G inside \mathcal{H}_3 , there is a subgraph \mathcal{H}' which is also universal triangle-free, and is coded by a strong coding tree S in which the coloring is monochromatic on all strictly similar copies in S of any fixed incremental strongly diagonal tree coding G . We already showed that among strongly diagonal trees coding G , it suffices to consider only those which are incremental, as this follows from Lemma 39. In this final section, we ensure that strongly diagonal trees are the only trees we need to consider coding copies of G , by showing that in any strong coding tree, there is an infinite subtree which is strongly diagonal and also codes \mathcal{H}_3 .

Theorem 45 (Strongly diagonal tree coding \mathcal{H}_3). *Let S be a strong coding tree. Then there is a strongly diagonal tree $D \subseteq S$ such that the terminal nodes in D are exactly the coding nodes in D and the coding nodes in D code \mathcal{H}_3 .*

Moreover, we construct D so that the following hold:

- (1) *The coding nodes in D code \mathcal{H}_3 in exactly the same way that S does: Letting c_n^D and c_n^S denote the n -th coding nodes in D and S , respectively, for all $i < n < \omega$, $c_n^D(|c_i^D|) = c_n^S(|c_i^S|)$.*
- (2) *For each $n < \omega$, the tree $D \upharpoonright (\leq |c_n^D| + 1) \setminus \{c_i^D : i \leq n\}$ is (disregarding coding nodes) tree isomorphic to the tree $S \upharpoonright (\leq |c_n^S| + 1)$.*

The proof of Theorem 45 will use the following Lemma.

Lemma 46. *Let T be any strong coding tree. Let $m < \omega$ be given and $t \in T \upharpoonright (l_m + 1)$ such that $t(l_m) = 1$, where l_m is the length of the m -th coding node t_m in T . (We are requiring $t(l_m) = 1$ here since we are going to be applying this lemma in the case where t is extended to the next coding node to be in D , so it will need to have a passing number of 1 at the previous coding node level.) Let S_{split} be the set of $s \in T \upharpoonright (l_m + 1)$ such that s and t have no parallel 1's, and let S_0 denote the set $T \upharpoonright (l_m + 1) \setminus (S_{\text{split}} \cup \{t\})$.*

Then there is an $n > m$ and extensions $t_ \supset t$, $s_*^0, s_*^1 \supset s$ for all $s \in S_{\text{split}}$ and $s_* \supset s$ for all $s \in S_0$, each of length $l_n + 1$, such that the following hold:*

- (1) *$t_* \upharpoonright l_n$ is a coding node.*
- (2) *$s_*^0(l_n) = 0$ and $s_*^1(l_n) = 1$ for each $s \in S_{\text{split}}$.*
- (3) *$s_*(l_n) = 0$ for each $s \in S_0$.*
- (4) *We may choose such extensions so that there are no parallel 1's between levels in $[l_m + 1, l_n]$.*
- (5) *All splitting happens between levels $l_{n-1} + 1$ and l_n in T , so that lexicographically larger nodes split at lower lengths than lexicographically smaller nodes. (That is, splitting goes from right to left, lower to higher.)*

Proof. Let t'_* be a coding node in T extending t such that $\text{lh}(t'_*) = l_n$ for some $n \geq m + 2$. Let t_* denote the immediate extension of t'_* in T ; note that t_* must be exactly $t'_* \hat{\ } 0$. Let S_{split} and S_0 be defined as in the hypotheses, and let $S = S_{\text{split}} \cup S_0$. For each $s \in S_0$, let s_* be the leftmost extension of s in T to level $l_n + 1$. Then $s_*(l_n) = 0$. Furthermore, s_* and t_* have parallel 1's only if s and t already had parallel 1's.

For each $s \in S_{\text{split}}$, let u_s be the leftmost extension of s in T to level $l_{n-1} + 1$. Then $u_s(l_{n-1}) = 0$. Furthermore, u_s and $t_* \upharpoonright (l_{n-1} + 1)$ have parallel 1's only if s and t already had parallel 1's. Note that for all pairs v, w in $\{s_* \upharpoonright (l_{n-1} + 1) : s \in S_0\} \cup \{u_s : s \in S_{\text{split}}\} \cup \{t_* \upharpoonright (l_{n-1} + 1)\}$, if v and w have parallel 1's, then $v \upharpoonright (l_m + 1)$ and $w \upharpoonright (l_m + 1)$ already had parallel 1's. In particular, the Parallel 1's Criterion is preserved. Then for each $s \in S_{\text{split}}$, since u_s and $t_* \upharpoonright (l_{n-1} + 1)$ have no parallel 1's, u_s splits exactly once between levels $[l_n + 1, l_n]$ to some extensions $s_*^0, s_*^1 \in T \upharpoonright l_n + 1$ such that $s_*^0(l_n) = 0$ and $s_*^1(l_n) = 1$. Note that $t_*(l_n) = 0$ so that these extensions do not have any parallel 1's with t_* .

Note also that between levels $l_{n-1} + 1$ and l_n , there are no new parallel 1's since T has no parallel 1's in these levels. Thus, the set $\{s_* : s \in S_0\} \cup \{s_*^i : s \in S_{\text{split}}, i < 2\} \cup \{t'_*\}$ is an extension to level $l_n + 1$ satisfying the lemma. \square

Remark. As is the property of \mathbb{T} and $S \leq T \leq \mathbb{T}$, we will be constructing D so that it has the properties that each $d_i(|d_{i-1}|) = 1$ and we split the node at level $l_i + 1$ that is going to

be extended to a coding node before we split any of the other nodes. So we will be applying the previous lemma above the level just above the split before we extend to the next coding node. Thus, if d_i is equal to t_k in T , then d_{i+1} is going to be t_m for some $m \geq k + 2$.

Proof of Theorem 45. We now construct D inside of the strong coding tree S from Theorem 44. First, we show how to construct a strongly diagonal tree $\mathbb{D} \subseteq \mathbb{T}$ which codes \mathcal{H}_3 and satisfies the theorem, since the indexing in \mathbb{T} is simpler than in S . Then, since S is strongly similar to \mathbb{T} , letting $\varphi : \mathbb{T} \rightarrow S$ be the strong similarity isomorphism between \mathbb{T} and S , the image $\varphi(\mathbb{D})$ will yield our strongly diagonal tree $D \subseteq S$ coding \mathcal{H}_3 . Throughout, we are tacitly using the Passing Number Choice Lemma 23 to ensure that we can extend nodes chosen so far to have the desired passing numbers at a desired coding node.

Recall that for each $i < \omega$, $\mathbb{T} \upharpoonright i$ has either a coding node or a splitting node. Let t_n denote the n -th coding node of \mathbb{T} and $l_n = |t_n|$. We shall let d_n denote the n -th coding node of \mathbb{D} and let k_n denote its length; j_n will be the index such that $d_n = t_{j_n}$, the j_n -th coding node in \mathbb{T} . Hence, k_n will simply equal l_{j_n} . The nodes of $\mathbb{D} \upharpoonright k_n \setminus \{d_n\}$ shall be indexed as $\{t_s : s \in \mathbb{T} \upharpoonright l_n\}$.

Let $\text{stem}(\mathbb{D}) = \text{stem}(\mathbb{T})$, which is simply $\langle \rangle$. Label this as $t_{\langle \rangle}$. Recall that $\text{Lev}_{\mathbb{T}}(1) = \{\langle 0 \rangle, \langle 1 \rangle\}$, and moreover, that $\langle 1 \rangle$ is extended in \mathbb{T} to the coding node $t_0 = \langle 1, 0 \rangle$, and $\langle 0 \rangle$ is split in \mathbb{T} to $\langle 0, 0 \rangle$ and $\langle 0, 1 \rangle$. $\text{Lev}_{\mathbb{T}}(l_0) = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle\}$. Recall also that $\{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle\}$ is exactly $\text{Lev}_{\mathbb{T}}(2)$. Extend $\langle 1 \rangle$ to a splitting node in \mathbb{T} and label it $t_{\langle 1 \rangle}^*$. Let m_0 be the index of the least coding node in \mathbb{T} longer than $t_{\langle 1 \rangle}^*$. Take $n_0 > m_0$ such that the coding node t_{n_0} extends $t_{\langle 1 \rangle}^* \frown 1$. Let $d_0 = t_{n_0}$; that is, the least coding node in \mathbb{D} is t_{n_0} . Extend $t_{\langle 1 \rangle}^* \frown 0$ leftmost to level l_{n_0} and label that node $t_{\langle 1, 0 \rangle}$. This is the node that will be extended in \mathbb{D} ; d_0 is a terminal node.

Next, extend $\langle 0 \rangle$ to $0^{l_{n_0-1}}$, the leftmost node in \mathbb{T} at level l_{n_0-1} , the length of the $n_0 - 1$ -st coding node in \mathbb{T} . By Lemma 46, $0^{l_{n_0-1}}$ extends to exactly one splitting node between the levels of l_{n_0-1} and l_{n_0} ; label this splitting node $t_{\langle 0 \rangle}^*$. Then for each $i < 2$, $t_{\langle 0 \rangle}^* \frown i$ extends in \mathbb{T} to a unique node at level l_{n_0} , label it $t_{\langle 0, i \rangle}$. Notice that $t_{\langle 0, i \rangle}$ extends precisely by i in \mathbb{T} .

Thus, we have $\{t_{\langle 0, 0 \rangle}, t_{\langle 0, 1 \rangle}, t_{\langle 1, 0 \rangle}, d_0\}$ at level l_{n_0} such that $\{t_{\langle 0, 0 \rangle}, t_{\langle 0, 1 \rangle}, t_{\langle 1, 0 \rangle}\}$ is strongly similar to $\mathbb{T} \upharpoonright l_0$ and d_0 is a coding node. We shall let $k_0 = l_{n_0}$ and

$$(43) \quad \mathbb{D} \upharpoonright k_0 = \{t_{\langle 0, 0 \rangle}, t_{\langle 0, 1 \rangle}, t_{\langle 1, 0 \rangle}, d_0\},$$

which is \mathbb{D} restricted to the level of the least coding node in \mathbb{D} . Further note that each member of $(\mathbb{D} \upharpoonright k_0) \setminus \{d_0\}$ extends uniquely in \mathbb{T} , and that this extension of $t_{\langle i_0, i_1 \rangle}$ has passing number i_1 as it passes d_0 . Let

$$(44) \quad \mathbb{D} \upharpoonright (k_0 + 1) = \{t_{\langle 0, 0 \rangle} \frown 0, t_{\langle 0, 1 \rangle} \frown 1, t_{\langle 1, 0 \rangle} \frown 0\},$$

the collection of immediate successors in \mathbb{T} of the non-coding nodes in level k_0 of \mathbb{D} .

For the induction step, suppose we have constructed $\mathbb{D} \upharpoonright (k_{n-1} + 1)$, where k_{n-1} is the length of the $(n - 1)$ -st coding node d_{n-1} in $\mathbb{D} \upharpoonright k_{n-1}$, and $(\mathbb{D} \upharpoonright (k_{n-1} + 1)) \setminus \{d_i : i < j\}$ is isomorphic as a tree to $\mathbb{T} \upharpoonright (l_{n-1} + 1)$, ignoring its coding nodes but taking into account the passing numbers at levels k_i and l_i , respectively.

For each $s \in \mathbb{T} \upharpoonright (l_{n-1} + 1)$, t_s denotes the member of $\mathbb{D} \upharpoonright k_{n-1} + 1$ such that $t_s(|d_i|) = s(|t_i|)$, for all $i < n$. Let s_* denote the member of $\mathbb{T} \upharpoonright (l_{n-1} + 1)$ which is the index of the node t_{s_*} in $\mathbb{D} \upharpoonright (k_{n-1} + 1)$ which needs to be extended to the next coding node d_n in $\mathbb{D} \upharpoonright k_n$. Let z denote t_{s_*} . Note that $z(l_{n-1})$ must equal 1.

Let z^* be a splitting node in \mathbb{T} extending z and let m_n be the least number such that $l_{m_n} > |z^*|$. Extend $z^* \frown 1$ to a coding node of length l_{j_n} for some $j_n > m_n$ and let $d_n = t_{j_n}$. Extend $z^* \frown 0$ via its leftmost extension in \mathbb{T} to level l_{j_n} and label this $t_{s_* \frown 1}$, because it has passing number 1 at d_{n-1} . Then the immediate extension of $t_{s_* \frown 1}$ in \mathbb{T} must be a 0.

For each $s \in \mathbb{T} \upharpoonright (l_{n-1} + 1) \setminus \{s_*\}$ such that s does not split in \mathbb{T} before level l_n , extend t_s via its leftmost extension in \mathbb{T} to level l_{j_n} and label it $t_{s \frown (0, \dots, 0)}$, where $(0, \dots, 0)$ is the sequence of 0's of length $l_{j_n} - (l_{n-1} + 1)$. Then the immediate extension in \mathbb{T} of each $t_{s \frown (0, \dots, 0)}$ is 0.

For each $s \in \mathbb{T} \upharpoonright (l_{n-1} + 1) \setminus \{s_*\}$ for which s has an extension in \mathbb{T} which splits in \mathbb{T} before level l_n , first extend t_s via its leftmost extension in \mathbb{T} to level l_{j_n-1} and call this t_s^* . Then there are exactly two extensions of t_s^* at level l_{j_n} , and these split in \mathbb{T} between the levels of $l_{j_n-1} + 1$ and l_{j_n} . Label these t_{s_i} , $i < 2$, where s_0 denotes the leftmost extension of s at level l_{j_n} and s_1 denotes the rightmost extension of s at level l_{j_n} . Note that the immediate extension of t_{s_i} in \mathbb{T} is i , for each $i < 2$. Let $k_n = l_{j_n}$. This constructs $\mathbb{D} \upharpoonright k_n$ according to our claims.

Let \mathbb{D}^- denote \mathbb{D} minus its coding nodes. At each stage n , we have

- (1) \mathbb{D} is strongly diagonal.
- (2) The terminal nodes in \mathbb{D} are exactly the set of coding nodes $\{d_n : n < \omega\}$.
- (3) $\mathbb{D}^- \cong \mathbb{T}$.
- (4) For each $i < n$, $d_n(k_i) = d_n^{\mathbb{T}}(|d_i^{\mathbb{T}}|)$; that is, the passing numbers of the d_n are exactly the same as the coding nodes in \mathbb{T} , and hence they code \mathcal{H}_3 in exactly the same order.
- (5) For each $n < \omega$, the nodes in the level of the n -th coding node d_n in \mathbb{D} are coded by $\text{Lev}_{\mathbb{T}}(n+2)$, with $t_s <_{\text{lex}} t_{s'}$ if and only if $s <_{\text{lex}} s'$, for $s, s' \in \text{Lev}_{\mathbb{T}}(n+2)$.

To finish, let f be the strong similarity map from \mathbb{T} onto S . Letting D be the f -image of \mathbb{D} , we obtain a strongly diagonal subtree of S . \square

Theorem 47 (The big Ramsey degrees for \mathcal{H}_3). *The universal triangle-free graph has finite big Ramsey degrees.*

Proof. Let G be a finite triangle-free graph, and let c be a coloring of all the copies of G in \mathcal{H}_3 into finitely many colors. The strong coding tree \mathbb{T} codes \mathcal{H}_3 . Let $S \leq \mathbb{T}$ be the tree from Theorem 44 so that for each incremental strongly diagonal tree Z_i coding G , c is monochromatic on all strictly similar copies of Z_i in S . By Lemma 45 there is a strongly diagonal subtree $D \subseteq S$ which also codes \mathcal{H}_3 . Then every subtree of D coding a copy of D is strongly diagonal. Therefore, the number of strict similarity types of incremental strongly diagonal trees coding G , one arrives at the big Ramsey degree of G in \mathcal{H}_3 yields an upper bound for the big Ramsey degrees. \square

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